

ACTION MINIMIZATION AND MACROSCOPIC INTERFACE MOTION UNDER FORCED DISPLACEMENT

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ABSTRACT. We study an one dimensional model where an interface is the stationary solution of a mesoscopic non local evolution equation which has been derived by a microscopic stochastic spin system. Deviations from this evolution equation can be quantified by obtaining the large deviations cost functional from the underlying stochastic process. For such a functional, derived in a companion paper, we investigate the optimal way for a macroscopic interface to move from an initial to a final position distant by R within fixed time T . We find that for small values of R/T the interface moves with a constant speed, while for larger values there appear nucleations of the other phase ahead of the front.

1. INTRODUCTION

In recent years, there has been a significant effort to derive deterministic models describing two-phase materials and their dynamical properties, [19]. Furthermore, with the inclusion of stochastic effects [17] one can study richer phenomena such as dynamic transitions between local minima. This is an extension of ideas already developed in the Freidlin-Wentzell theory [18] on random perturbation of dynamical systems. Such effects, can be encoded to action functionals whose minimizers prescribe the optimal transition. The choice of the action functional is not straightforward. The purpose of this paper and of the companion [8], is to show that given the *mesoscopic* deterministic partial differential equation (PDE), one can consider the underlying *microscopic* stochastic process (whose scaling limit is the given PDE) and calculate the corresponding large deviations functional which would provide the action functional we are after. This is a well developed idea also in the more general setting of nonequilibrium systems [6] and here we examine it in the context of reversible dynamics describing *macroscopic* interface motion. Furthermore, this connection to the underlying stochastic process is also insightful for calculating the minimizers. For example, in the present work we borrow concepts from statistical mechanics such as contours, free energy, local equilibrium which allow us to better understand the structure of the cost functional and hence reduce it in a simpler and more easily treatable form.

Similar results have been obtained in the context of the stochastic Allen-Cahn equation. In [20, 21] the authors study the same problem for $d = 1$ while in [23, 5] it is extended to $d = 2, 3$. In particular, in [5] the limit considered is a joint sharp interface and small noise, but the starting point is at the mesoscopic scale, even though noise is also involved. Some

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numerical results were also presented in [16]. In this context, our contribution in this and the companion paper [8] is that we derive (and subsequently minimize) the large deviations action functional directly from a microscopic process, hence completing this program of connecting the three scales: microscopic (process), mesoscopic (equation) and macroscopic (sharp-interface). However, for technical reasons we have to restrict ourselves in $d = 1$ even though several partial results are valid also in higher dimensions. Note also that in a coarse-grained (almost mesoscopic) scale, we have an equation which is comparable to a non-local Allen-Cahn type equation with a noise which is a martingale generated by the microscopic noise of each spin. On the other hand, in the stochastic Allen-Cahn one adds by hand a “mesoscopic” white-noise in one dimension, or a properly coloured noise in higher dimensions (for more details about the motivation see the introduction in [4]). The connection to the stochastic Allen-Cahn is particularly interesting also in view of the results [7, 22] connecting the fluctuations of this microscopic process to the stochastic Allen-Cahn equation in a critical regime. We conclude mentioning that the meso-to-macro limit for a closely related evolution equation has been already addressed in [11], but for a postulated action functional given by the L^2 norm of an external force corresponding to the deviating profiles. In fact, we show that the large deviations functional gives a softer penalization on deviating profiles than the L^2 norm considered in [11], hence our task here is a bit harder and we need to properly adjust the proof of [11] in the new context.

2. THE MODEL AND THE MAIN RESULT

We work in the context of a nonlocal evolution equation which can be derived by an interacting particles system of Ising spins with Kac interaction and Glauber dynamics, [10] and [13]:

$$\frac{d}{dt}m = -m + \tanh\{\beta(J * m)\}, \quad m(0, x) = m_0(x), \quad (2.1)$$

where $J * m(x, t) = \int_{\mathbb{R}} J(x - y)m(y, t) dy$ and $J \in C^2(\mathbb{R})$ is even, $J(r) = 0$ for all $|r| > 1$, $\int_{\mathbb{R}} J(r)dr = 1$ and non increasing for $r > 0$. We also suppose $\beta > 1$. Furthermore, this equation is related to the gradient flow of the free energy functional

$$\mathcal{F}(m) = \int_{\mathbb{R}} \phi_{\beta}(m)dx + \frac{1}{4} \int_{\mathbb{R} \times \mathbb{R}} J(x, y)[m(x) - m(y)]^2 dx dy, \quad (2.2)$$

where $\phi_{\beta}(m)$ is the “mean field excess free energy”

$$\phi_{\beta}(m) = \tilde{\phi}_{\beta}(m) - \min_{|s| \leq 1} \tilde{\phi}_{\beta}(s), \quad \tilde{\phi}_{\beta}(m) = -\frac{m^2}{2} - \frac{1}{\beta} \mathcal{S}(m), \quad \beta > 1,$$

and $\mathcal{S}(m)$ the entropy:

$$\mathcal{S}(m) = -\frac{1-m}{2} \log \frac{1-m}{2} - \frac{1+m}{2} \log \frac{1+m}{2}.$$

We also denote by

$$f(m) := \frac{\delta \mathcal{F}}{\delta m} = -J * m + \frac{1}{\beta} \operatorname{arctanh} m \quad (2.3)$$

the functional derivative of \mathcal{F} . Thus, the functional in (2.2) is a Lyapunov functional for the equation (2.1):

$$\frac{d}{dt} \mathcal{F}(m) = -\frac{1}{\beta} \int_{\mathbb{R}} (-\beta J * m + \operatorname{arctanh} m)(m - \tanh(\beta J * m)) dx \leq 0,$$

since the two factors inside the integral have the same sign. This structure will be essential in the sequel, e.g. in Theorem 2.1.

Concerning the stationary solutions of the equation (2.1) in \mathbb{R} , it has been proved that the two constant functions $m^{(\pm)}(x) := \pm m_\beta$, with $m_\beta > 0$ solving the mean field equation $m_\beta = \tanh\{\beta m_\beta\}$ are stationary solutions of (2.1) and are interpreted as the two pure phases of the system with positive and negative magnetization.

Interfaces, which are the objects of this paper, are made up from particular stationary solutions of (2.1). Such solutions, called *instantons*, exist for any $\beta > 1$ and we denote them by $\bar{m}_\xi(x)$, where ξ is a parameter called the center of the instanton. Denoting $\bar{m} := \bar{m}_0$, we have that

$$\bar{m}_\xi(x) = \bar{m}(x - \xi), \quad (2.4)$$

where the instanton \bar{m} satisfies

$$\bar{m}(x) = \tanh\{\beta J * \bar{m}(x)\}, \quad x \in \mathbb{R}. \quad (2.5)$$

It is an increasing, antisymmetric function which converges exponentially fast to $\pm m_\beta$ as $x \rightarrow \pm\infty$, see e.g. [14], and there are α and a positive so that

$$\lim_{x \rightarrow \infty} e^{\alpha x} \bar{m}'(x) = a, \quad (2.6)$$

see [12], Theorem 3.1. Moreover, any other solution of (2.5) which is strictly positive [respectively negative] as $x \rightarrow \infty$ [respectively $x \rightarrow -\infty$], is a translate of $\bar{m}(x)$, see [15]. Note also that in the case of finite volume $[-\epsilon^{-1}L, \epsilon^{-1}L]$ the solution $\bar{m}^{(\epsilon)}$ with Neumann boundary conditions is close to \bar{m} : for every $\epsilon > 0$ we consider the non-local mean field equation

$$m^{(\epsilon)} = \tanh\{\beta J^{\text{neum}} \star m^{(\epsilon)}\}, \quad |x| \leq \epsilon^{-1}L, \quad (2.7)$$

where $m^{(\epsilon)} \in L^\infty([-\epsilon^{-1}L, \epsilon^{-1}L]; [-1, 1])$ and

$$J^{\text{neum}}(x, y) := J(x, y) + J(x, R_{\epsilon^{-1}L}(y)) + J(x, R_{-\epsilon^{-1}L}(y)),$$

with $R_l(y) := l - (y - l)$ being the reflection of y around l . By following [3], Section 3, or [1], Section 3.3, given $\zeta > 0$ there exists ϵ_0 such that for every $\epsilon < \epsilon_0$, there is $\bar{m}^{(\epsilon)}$ which is antisymmetric, solves (2.7), satisfies

$$\|\bar{m}^{(\epsilon)} - \bar{m}\|_{L^\infty([-\epsilon^{-1}L, \epsilon^{-1}L])} < \zeta \quad (2.8)$$

and it is unique in the above neighbourhood. See also [24], section 6.2.3.

Hence, if we start with an instanton, the evolution (2.1) will not move it. So, in order to impose a speed to the interface one has to add an external force to the equation (2.1). The

result would be a deviation from (2.1) and any such deviation $\{\phi(x, t)\}_{x, t}$ corresponds to an external force that can produce it and which is given by

$$b(\phi)(x, t) := \dot{\phi}(x, t) + \phi(x, t) - \tanh(\beta J * \phi(x, t)), \quad (2.9)$$

where we have introduced the notation $\dot{\phi}(x, t) := \frac{d}{dt}\phi(x, t)$ and for b we explicit the dependence on ϕ . Later, when this dependence is not relevant we will only use b . Thus, such deviating profiles can be viewed as solutions of the following forced equation:

$$\frac{d}{dt}m = -m + \tanh(\beta J * m) + b, \quad m(x, 0) = m_0(x), \quad (2.10)$$

where the force term b is some prescribed function of x and t . In this paper, we are interested in investigating the response of the system when imposing a mean velocity V to the front, i.e., we want to displace the interface from an initial position 0 to a final one, R , within a fixed time $T = R/V$. We consider two scales: the mesoscopic where the interface is diffuse and the macroscopic where the interface has a sharp jump, i.e., it is given by the step function $m_\beta(\mathbf{1}_{x \geq 0} - \mathbf{1}_{x < 0})$. Let $[0, T] \times \mathbb{R}$ be the macroscopic time-space domain. After rescaling back to the mesoscopic variables we are interested in profiles in the set $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ where

$$\mathcal{U}[r, t] = \{\phi \in C^\infty(\mathbb{R} \times (0, t); (-1, 1)) : \lim_{s \rightarrow 0^+} \phi(\cdot, s) = \bar{m}, \lim_{s \rightarrow t^-} \phi(\cdot, s) = \bar{m}_r\} \quad (2.11)$$

and where now in the mesoscopic variables the fronts are represented by the instantons \bar{m} and \bar{m}_r . Due to the stationarity of \bar{m} , no element in $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ is a solution to the equation (2.1). Instead, to each element in $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ it corresponds an external force b as in (2.9), and in order to select among such forces one needs to introduce an appropriate action functional. In [11], the authors invoking linear response theory suggested the cost functional to be given by $\int_0^{\epsilon^{-2}T} \int_{\mathbb{R}} b(x, t)^2 dx dt$. In a companion paper, [8], instead of postulating the cost, we derive it directly from the underlying stochastic mechanism via large deviations over a certain class of functions. More precisely, to derive the cost from the stochastic dynamics we work in the space domain $[-\epsilon^{-1}L, \epsilon^{-1}L] \subset \mathbb{R}$ with Neumann boundary conditions. As it will be shown later, the main objects to which the cost concentrates are the instantons, which decay exponentially fast as $x \rightarrow \pm\infty$ and are well approximated by their finite volume counterparts as in (2.8). Hence, in order to avoid unnecessary technical complications we can concentrate here in the whole \mathbb{R} and denote the new cost on $\mathbb{R} \times [0, \epsilon^{-2}T]$ by:

$$I_{[0, \epsilon^{-2}T]}(\phi) = \int_0^{\epsilon^{-2}T} \int_{\mathbb{R}} \mathcal{H}(\phi, \dot{\phi})(x, t) dx dt, \quad (2.12)$$

where for notational simplicity we neglect the dependence of the cost on \mathbb{R} . The density $\mathcal{H}(\phi, \dot{\phi})$ is given below and we will also denote it by $\mathcal{H}(x, t)$ in case we do not need to explicit the dependence on ϕ . Given $(\phi, \dot{\phi})$ we define

$$\begin{aligned} u &:= \phi \\ w &:= -\tanh(\beta J * \phi) \\ b &:= \dot{\phi} + \phi - \tanh(\beta J * \phi) \end{aligned}$$

and after a simple manipulation by a small abuse of notation we can write \mathcal{H} as depending on (b, u, w) in the following form:

$$\begin{aligned} \mathcal{H}(b, u, w) = & \frac{1}{2} \left\{ (b - u - w) \log \frac{b - u - w + \sqrt{(b - u - w)^2 + (1 - u^2)(1 - w^2)}}{(1 - u)(1 - w)} \right. \\ & \left. - \sqrt{(b - u - w)^2 + (1 - u^2)(1 - w^2)} + 1 + uw \right\}. \end{aligned} \quad (2.13)$$

The new functional, has a more complicated structure, but asymptotically has a similar behaviour: It is a straightforward calculation to see that uniformly on $u \in [-1, 1]$ and $w \in (-1, 1)$ we have:

$$\lim_{|b| \rightarrow \infty} \frac{\mathcal{H}(b, u, w)}{|b| \log(|b| + 1)} = \frac{1}{2} \quad \text{and} \quad \lim_{|b| \rightarrow 0} \frac{\mathcal{H}(b, u, w)}{b^2} = \frac{1}{4(1 + uw)}. \quad (2.14)$$

Note that the cost assumed in [11] is approximating the case when b is small, but when b is large they are far from each other; hence it gives a stronger penalization of the deviating profiles than the one derived from the microscopic system. As we shall also see in the sequel, the minimizers will correspond to external fields b which are ϵ -small, so it is expected that the minimizers of the new functional will be the same with [11]. But still, we can not exclude a priori the cases that correspond to large external fields and this is a technical difficulty we have to overcome. Furthermore, we have a slightly different equation and a more complicated form of the cost. Thus, in this paper, we find the minimizer of the derived cost $I_{[0, \epsilon^{-2}T]}(\phi)$ given in (2.12) over the class (2.11) following the strategy in [11] and adjusting the proof accordingly in order to overcome the aforementioned technical issues. To start with, we observe that the cost of a moving instanton with ϵ -small velocity, i.e.,

$$\phi_\epsilon(x, t) = \bar{m}_{\epsilon V t}(x), \quad V = \frac{R}{T},$$

is given by

$$I_{[0, \epsilon^{-2}T]}(\phi_\epsilon) = \frac{1}{4} \|\bar{m}'\|_{L^2(d\nu)}^2 V^2 T,$$

where \bar{m}' is the derivative of \bar{m} and $\|\cdot\|_{L^2(d\nu)}$ denotes the L^2 norm on $(\mathbb{R}, d\nu(x))$ with $d\nu(x) = \frac{dx}{1 - \bar{m}^2(x)}$. As in [11] it can be shown that other ways to move continuously the instanton are more expensive.

In such systems one can also observe the phenomenon of nucleations, namely the appearance of droplets of a phase inside another. In [1] and [2] it has been proved that for such a profile the cost is bounded by twice the free energy computed at the instanton:

Theorem 2.1. *For any $\vartheta > 0$ there is $\tau > 0$ and a function $\tilde{m}_{\epsilon, \tau}(x, s)$, $x \in \mathbb{R}$, $s \in [0, \tau \epsilon^{-3/2}]$, symmetric in x for each s and such that*

$$\tilde{m}_{\epsilon, \tau}(x, 0) = m_\beta, \quad \tilde{m}_{\epsilon, \tau}(x, \tau \epsilon^{-3/2}) = \bar{m}_{\ell_\epsilon/2}(x), \quad x \geq 0, \quad (2.15)$$

where $e^{-\alpha \ell_\epsilon} = \epsilon^{3/2}$, $\alpha > 0$ as in (2.6), and

$$I_{\tau \epsilon^{-3/2}}(\tilde{m}_{\epsilon, \tau}) \leq 2\mathcal{F}(\bar{m}) + \vartheta. \quad (2.16)$$

Thus, if V gets large, there is a competition between the two values of the cost. Therefore, by creating more fronts we can make them move with smaller velocity with the gain in cost being larger than the extra penalty for the nucleations. Following [11] we define:

$$w_n(R, T) := n2\mathcal{F}(\bar{m}) + (2n + 1) \left\{ \frac{1}{\mu} \left(\frac{V}{2n + 1} \right)^2 T \right\}, \quad (2.17)$$

where $\mu =: 4\|\bar{m}'\|_{L^2(d\nu)}$ is the mobility coefficient. The first term is the cost of n nucleations while the second is the cost of displacement of $2n + 1$ fronts (with the smaller velocity $V/(2n + 1)$). Our main result is given below:

Theorem 2.2. *Let $P > \inf_{n \geq 0} w_n(R, T)$.*

- (i) *Then $\forall \gamma > 0$ there exists an $\epsilon_1 > 0$ such that $\forall \epsilon < \epsilon_1$ and for all sequences $\phi_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ with*

$$I_{\Lambda_\epsilon \times \mathcal{T}_\epsilon}(\phi_\epsilon) \leq P, \quad (2.18)$$

we have:

$$I_{\Lambda_\epsilon \times \mathcal{T}_\epsilon}(\phi_\epsilon) \geq \inf_{n \geq 0} w_n(R, T) - \gamma, \quad (2.19)$$

where $w_n(R, T)$ is given in (2.17).

- (ii) *There exists a sequence $\phi_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ such that*

$$\limsup_{\epsilon \rightarrow 0} I_{\Lambda_\epsilon \times \mathcal{T}_\epsilon}(\phi_\epsilon) \leq \inf_{n \geq 0} w_n(R, T). \quad (2.20)$$

We split the proof in the following sections: in Section 3 we first recall the notions of contours that allow us to separate the phases. Then we present the multi-instanton manifold and its properties. This is a repetition of [11] and the reader familiar with it could skip it. However, for completeness of the presentation we also include it here as we will need several of these concepts in the next sections. One of the key estimates in the proof is the fact that, because of the finite cost, the profiles can not be away from local equilibrium (instanton manifold) for too long as there is a driving gradient force pushing them back. The main ingredients for this are given in Section 4 and the key Proposition 4.4 is a bit different than [11], so its proof is adjusted to the new context. In Section 5 we outline the proof which consists in splitting the time into good/bad time intervals during which the cost is small/large, respectively. Moreover, we establish the fact that we cannot stay away from the instanton manifold for too long as the gradient dynamics drive us back. Hence, in good time intervals we will eventually find ourselves close to the instanton manifold and, once this happens, we stay there for the whole interval. Then, we can linearize around some instanton and attribute some velocity to each interface. This is presented in Section 6. Furthermore, we still need to “connect” the good time intervals between them and this will be explained in Section 7. On the other hand, during bad time intervals which are treated in Section 8, more interesting things can happen, namely creation of new fronts (nucleations). But due to the fact that the overall cost is finite, they cannot be too many and the overall displacement during the bad time intervals is negligible. Concluding, having split the cost into smooth displacement (with some velocity) and nucleations, we introduce a simplified, closer to macroscopic, model

for the motion of the “centers” of the instantons. We call it “particle model” and analyze it in Section 9 concluding the proof of Theorem 2.2. Some further technical issues are left for the Appendix.

3. PRELIMINARIES

In this section we recall some facts that we will use in the sequel. For a more complete exposition we refer the reader to the original paper [11] and to the monograph [24]. We start with the definition of contours and the Peierls estimates which are bounds on the spatial location of deviations from the equilibrium in terms of the energy \mathcal{F} .

3.1. Contours. Given $\ell > 0$, we denote by $\mathcal{D}^{(\ell)}$ the partition of \mathbb{R} into the intervals $[n\ell, (n+1)\ell)$, $n \in \mathbb{Z}$, and by $Q_x^{(\ell)}$, $x \in \mathbb{R}$ the interval containing x (note that x need not be the center of $Q_x^{(\ell)}$). We say that $Q_x^{(\ell)}$, $Q_{x'}^{(\ell)}$ are connected, if the closures have nonempty intersection, i.e. $\overline{Q_x^{(\ell)}} \cap \overline{Q_{x'}^{(\ell)}} \neq \emptyset$. Now we define

$$m^{(\ell)}(x) := \frac{1}{|Q_x^{(\ell)}|} \int_{Q_x^{(\ell)}} m(y) dy. \quad (3.1)$$

Given an “accuracy parameter” $\zeta > 0$, we introduce

$$\eta^{(\zeta, \ell)}(m; x) = \begin{cases} \pm 1 & \text{if } |m^{(\ell)}(x) \mp m_\beta| \leq \zeta, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

For any $\Lambda \subseteq \mathbb{R}$ which is $D^{(\ell)}$ -measurable we call

$$\begin{aligned} \mathcal{B}_0^{(\zeta, \ell, \Lambda)}(m) &:= \{x \in \Lambda : \eta^{(\zeta, \ell)}(m; x) = 0\} \\ \mathcal{B}_\pm^{(\zeta, \ell, \Lambda)}(m) &:= \left\{x \in \Lambda : \eta^{(\zeta, \ell)}(m; x') = \pm 1, \text{ there exists } x' \in \Lambda : \overline{Q_x^{(\ell)}} \cap \overline{Q_{x'}^{(\ell)}} \neq \emptyset \right. \\ &\quad \left. \eta^{(\zeta, \ell)}(m; x') = -\eta^{(\zeta, \ell)}(m; x) \right\}, \\ \mathcal{B}_0^{(\zeta, \ell, \Lambda)}(m) &:= \mathcal{B}_+^{(\zeta, \ell, \Lambda)}(m) \cup \mathcal{B}_-^{(\zeta, \ell, \Lambda)}(m) \cup \mathcal{B}_0^{(\zeta, \ell, \Lambda)}(m). \end{aligned}$$

Calling ℓ_- and ℓ_+ two values of the parameter ℓ , with ℓ_+ an integer multiple of ℓ_- , we define a “phase indicator”

$$\vartheta^{(\zeta, \ell_-, \ell_+)}(m; x) = \begin{cases} \pm 1 & \text{if } \eta^{(\zeta, \ell_-)}(m; \cdot) = \pm 1 \text{ in } \left(Q_{x-\ell_+}^{(\ell_+)} \cup Q_x^{(\ell_+)} \cup Q_{x+\ell_+}^{(\ell_+)} \right), \\ 0 & \text{otherwise,} \end{cases}$$

and call contours of m the connected components of the set $\{x : \vartheta^{(\zeta, \ell_-, \ell_+)}(m; x) = 0\}$. The interval $\Gamma = [x_-, x_+]$ is a plus contour if $\eta^{(\zeta, \ell_-)}(m; x_\pm) = 1$, a minus contour if $\eta^{(\zeta, \ell_-)}(m; x_\pm) = -1$, otherwise it is called mixed.

Moreover, for any measurable $\Lambda \subseteq \mathbb{R}$ and $m \in L^\infty(\mathbb{R} \rightarrow [-1, 1])$, we define a local notion of energy by

$$\begin{aligned} \mathcal{F}(m_\Lambda | m_{\Lambda^c}) &:= \int_\Lambda \phi_\beta(x) dx + \frac{1}{4} \int_{\Lambda \times \Lambda} J(x, y) (m(x) - m(y))^2 dy dx \\ &\quad + \frac{1}{2} \int_{\Lambda \times \Lambda^c} J(x, y) (m(x) - m(y))^2 dy dx. \end{aligned}$$

The parameters (ζ, ℓ_-, ℓ_+) are called compatible with $(\zeta_0, c_1, \kappa) \in \mathbb{R}_+^3$ if $\zeta \in (0, \zeta_0)$, $\ell_- \leq \kappa \zeta$, $\ell_+ \geq 1/\ell_-$, and if for any $D^{(\ell_-)}$ -measurable set Λ and any $m \in L^\infty(\mathbb{R} \rightarrow [-1, 1])$

$$\mathcal{F}(m_\Lambda | m_{\Lambda^c}) \geq c_1 \zeta^2 |\mathcal{B}_0^{(\zeta, \ell_-, \Lambda)}(m)|.$$

With the above definitions we have:

Theorem 3.1 ([1]). *There are positive constants $\zeta_0, c_1, \kappa, c_2$, and α so that if (ζ, ℓ_-, ℓ_+) is compatible with (ζ_0, c_1, κ) , then for all $m \in L^\infty([-L, L]; [-1, 1])$,*

$$\mathcal{F}(m) \geq \sum_{\Gamma \text{ contour of } m} w_{\zeta, \ell_-, \ell_+}(\Gamma), \quad (3.3)$$

where

$$\begin{aligned} w_{\zeta, \ell_-, \ell_+}(\Gamma) &= c_1 \zeta^2 \frac{\ell_-}{\ell_+} |\Gamma| \text{ if } \Gamma \text{ is a plus or a minus contour;} \\ w_{\zeta, \ell_-, \ell_+}(\Gamma) &= \max \left\{ c_1 \zeta^2 \frac{\ell_-}{\ell_+} |\Gamma| ; \mathcal{F}(\bar{m}) - c_2 e^{-\alpha \ell_+} \right\} \text{ if } \Gamma \text{ is a mixed contour.} \end{aligned}$$

Then, by Theorem 3.1, for ζ small enough,

$$\sum_{\Gamma_i \text{ contours of } u(\cdot, t)} |\Gamma_i| \leq \frac{\ell_+}{c_1 \ell_-} \zeta^{-2} (P + F(\bar{m})) \quad (3.4)$$

$$\text{number of contours of } u(\cdot, t) \leq \frac{1}{c_1 \ell_-} \zeta^{-2} (P + F(\bar{m})) =: N_{\max} \quad (3.5)$$

$$\text{number of mixed of contours of } u(\cdot, t) \leq \frac{P + F(\bar{m})}{\mathcal{F}(\bar{m}) - c_2 e^{-\alpha \ell_+}} =: N_{\max}^{\text{mix}} \quad (3.6)$$

3.2. Multi-instanton manifold. The instanton manifold is the set $\mathcal{M}^{(1)} = \{\bar{m}_\xi, \xi \in \mathbb{R}\}$. We extend the notion to the case of several coexisting instantons by defining the multi-instanton manifold $\mathcal{M}^{(k)}$, $k > 1$, as the set of all $\bar{m}_{\bar{\xi}}$, $\bar{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$, $\xi_1 < \dots < \xi_k$, sufficiently apart from each other such that, setting $\xi_0 := -\infty$, $\xi_{k+1} := \infty$, the function

$$\bar{m}_{\bar{\xi}}(x) := \begin{cases} \bar{m}(x - \xi_j) & \text{if } x \in \left[\frac{\xi_{j-1} + \xi_j}{2}, \frac{\xi_{j+1} + \xi_j}{2} \right] \text{ and } j \text{ odd,} \\ \bar{m}(\xi_j - x) & \text{if } x \in \left[\frac{\xi_{j-1} + \xi_j}{2}, \frac{\xi_{j+1} + \xi_j}{2} \right] \text{ and } j \text{ even,} \end{cases}$$

has exactly k mixed contours. We denote

$$\mathcal{M} = \bigsqcup_{k \geq 1} \mathcal{M}^{(k)}. \quad (3.7)$$

To study “neighborhoods” of \mathcal{M} we introduce the notion of “center of m ” that we use here in a slightly different sense than usual:

Definition 3.2. *Recalling $L^2(d\nu_\xi)$, the point $\xi \in \mathbb{R}$ is a center of m if $\xi \in \Gamma$, Γ a mixed contour of m , and if*

$$(m - \bar{m}_\xi, \bar{m}'_\xi)_{L^2(d\nu_\xi)} = 0, \quad \text{or, equivalently,} \quad (m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} = 0. \quad (3.8)$$

ξ is an odd, even, center if Γ is a $(-, +)$, respectively $(+, -)$ mixed contour.

The following theorem holds, see [14],

Theorem 3.3. *If ζ (in the definition of contours) is small enough the following holds.*

- *Each mixed contour Γ of m contains a center of m .*
- *There is $\delta > 0$ so that if for some ξ in a $(-, +)$ mixed contour Γ of m (analogous statement holding in the $(+, -)$ case), $\|\mathbf{1}_\Gamma(m - \bar{m}_\xi)\|_{L^2(d\nu_\xi)} \leq \delta$, then there is a unique center ξ_m in Γ and*

$$\int_{\mathbb{R}} \left(\{m - \bar{m}_{\xi'}\}^2 - \{m - \bar{m}_{\xi_m}\}^2 \right) > 0, \quad \text{for all } \xi' \in \Gamma, \xi' \neq \xi_m \quad (3.9)$$

and calling $v = m - \bar{m}_\xi$, $N_{v,\xi} = \frac{(v, \bar{m}'_\xi)}{(\bar{m}', \bar{m}')} ,$

$$|\xi_m - (\xi - N_{v,\xi})| \leq c\|v\|_{L^2(d\nu_\xi)}^2, \quad |N_{v,\xi}| \leq c\|v\|_{L^2(d\nu_\xi)}. \quad (3.10)$$

- *If also $\inf_{\xi} \|\mathbf{1}_\Gamma(m - \bar{m}_{\xi_n})\|_{L^2(d\nu_\xi)} \leq \delta$, then*

$$|\xi_m - \xi_n| \leq c\|m - n\|_{L^2(d\nu_\xi)}. \quad (3.11)$$

In Appendix B we will prove the third statement for both the L^1 and the L^2 norm. By the first statement in Theorem 3.3 a function m with k mixed contours $\Gamma_1, \dots, \Gamma_k$ has (at least) one center in each one of the mixed contours; we denote by Ξ the collection of all $\bar{\xi} = (\xi_1, \dots, \xi_k)$, $\xi_i < \xi_{i+1}$, ξ_i a center of m in Γ_i and define

$$d_{\mathcal{M}}(m) = \inf_{\bar{\xi} \in \Xi} \|m - \bar{m}_{\bar{\xi}}\|_{L^2(d\nu_{\bar{\xi}})}. \quad (3.12)$$

If m is close enough to $\mathcal{M}^{(k)}$, then the choice of $\bar{\xi}$ is unique. Note that this definition differs slightly from the usual definition of a distance of a point from a manifold, but the following lemma bounds this difference by replacing the inf over centers in (3.12), by the inf over any generic variable $\bar{\xi} \in \Gamma_1 \times \dots \times \Gamma_k$, with $\bar{\xi} = (\xi_1, \dots, \xi_k)$:

Lemma 3.4. *For all $k \in \mathbb{N}$ there are $\delta > 0$ and $c > 0$ so that if m has k mixed contours $\Gamma_1, \dots, \Gamma_k$ and $d_{\mathcal{M}}(m) \leq \delta$, then*

$$d_{\mathcal{M}}^2(m) \geq \inf_{\bar{\xi} \in \Gamma_1 \times \dots \times \Gamma_k} \|m - \bar{m}_{\bar{\xi}}\|_{L^2(d\nu_{\bar{\xi}})}^2 \geq d_{\mathcal{M}}^2(m) - c \sum_{i=1}^{k-1} e^{-\alpha \text{dist}(\Gamma_{i+1}, \Gamma_i)/2}, \quad (3.13)$$

where $\alpha > 0$ is defined in (2.6).

For the proof we refer to [11].

4. PERMANENCE AWAY FROM EQUILIBRIUM

In this section we get bounds on the time interval when a profile is away from the multi-stanton manifold. This is done by obtaining a lower bound on the energy gradient in terms of the distance from the manifold and we will use it in Theorem 5.4 in order to get a bound on the number of time intervals where the given profile is away from local equilibrium. The main theorem is:

Theorem 4.1. *For any $\vartheta > 0$ there is $\rho > 0$ such that the following holds. Let $m \in L^\infty(\mathbb{R}; (-1, 1))$ have an odd number p of mixed contours, let $\mathcal{F}(m) \leq P$ (P as in Theorem 2.2) and let $d_{\mathcal{M}}(m)^2 \geq \vartheta$. Then*

$$\int_{\mathbb{R}} (1 \wedge |f(m)|)^2 \geq \rho, \quad (4.1)$$

where f is defined in (2.3).

The proof is essentially contained in [11]. Here we only present the necessary modifications needed for the new functional. This theorem implies a penalization of the time away local equilibrium which is stated in the following corollary:

Corollary 4.2. *Let ϕ satisfy (2.18), then for any $\vartheta > 0$ there is $c_{4.2} > 0$ and $\rho > 0$ so that, if $d_{\mathcal{M}}(\phi(\cdot, t)) \geq \vartheta$ when $t \in [t_0, t_1]$, $0 \leq t_0 < t_1 \leq \epsilon^{-2}T$, then necessarily $t_1 - t_0 \leq \frac{3P}{c_{4.2}\rho}$.*

Proof. From [9] we have that:

$$I_{[t_0, t_1]}(\phi) \geq \frac{\beta}{2}(\mathcal{F}(\phi(\cdot, t_1)) - \mathcal{F}(\phi(\cdot, t_0))) + \int_{t_0}^{t_1} \|1 \wedge |f(\phi)|\|_2^2 dt$$

and since

$$\sup_{t \leq \epsilon^{-2}T} (\mathcal{F}(\phi(\cdot, t)) - \mathcal{F}(\phi(\cdot, 0))) \leq P,$$

from Theorem 4.1 we obtain that for some $c_{4.2} > 0$

$$3P \geq c_{4.2} \int_{t_0}^{t_1} \|1 \wedge |f(\phi)|\|_2^2 dt \geq c_{4.2}\rho(t_1 - t_0),$$

which concludes the proof. \square

Now we argue as in [11]. We start with the analysis of the condition $d_{\mathcal{M}}(m)^2 \geq \vartheta$ when the deviation of m from \bar{m}_{ξ} is localized in a neighborhood of the contours. We first give the necessary notation. Let Q , Q_j and $B_{k,j}^\pm$ be intervals of the form $Q = [a, b)$, $Q_j = [a - j, b + j)$, $B_{k,j}^- = [a - j - k, a - j)$, $B_{k,j}^+ = [b + j, b + j + k)$ with a, b, j, k all in $\ell_+\mathbb{N}$. Then, given $\vartheta > 0$, we set

$$U_{Q,j,\vartheta} = \left\{ m \in L^\infty(\mathbb{R}, (-1, 1)) : Q \text{ is a mixed } \pm \text{ contour for } m \text{ and } \inf_{\xi \in Q} \int_{Q_j} |m - \bar{m}_{\xi}|^2 \geq \vartheta \right\} \quad (4.2)$$

and

$$V_{k,j} = \left\{ m \in L^\infty(\mathbb{R}, (-1, 1)) : \eta^{(\zeta, \ell_-)}(m; x) = \pm 1 \text{ for all } x \in B_{k,j}^\pm \right\}. \quad (4.3)$$

Lemma 4.3. *For any $\vartheta > 0$, Q and Q_j as above, there is k so that*

$$\int_{Q_{k+j}} |f(m)| > 0 \quad \text{for any } m \in U_{Q,j,\vartheta} \cap V_{k,j}. \quad (4.4)$$

The proof is given in [11]. With this lemma we can prove the following:

Proposition 4.4. *For any $\vartheta > 0$, Q and Q_j , let k be as in Lemma 4.3. Then there is $\rho > 0$ so that*

$$\inf_{m \in U_{Q,j,\vartheta} \cap V_{k,j}} \int_{Q_{k+j}} |1 \wedge |f(m)||^2 \geq \rho. \quad (4.5)$$

Proof. Suppose that the opposite is true. Then there exists a sequence $m_n \in U_{Q,j,\vartheta} \cap V_{k,j}$ such that

$$\lim_{n \rightarrow \infty} \int_{Q_{k+j}} |1 \wedge |f(m_n)||^2 = 0,$$

which implies that $|A_n^c| \rightarrow 0$ and $\int_{Q_{k+j} \cap A_n} |f(m_n)|^2 \rightarrow 0$ where $A_n := \{x : |f(m_n(x))| < 1\}$. We also have that $m_n \rightharpoonup \hat{m}$ in L_{loc}^2 and hence $J * m_n \rightarrow J * \hat{m}$ in L_{loc}^2 . We write (recall that $f(m) = J * m - \text{arctanh } m$):

$$\begin{aligned} m_n &= m_n \mathbf{1}_{A_n} + m_n \mathbf{1}_{A_n^c} = \tanh(J * (m_n \mathbf{1}_{A_n}) - f(m_n \mathbf{1}_{A_n})) \mathbf{1}_{A_n} + m_n \mathbf{1}_{A_n^c} \\ &= \tanh(\beta J * m_n - f(m_n) \mathbf{1}_{A_n}) \mathbf{1}_{A_n} + m_n \mathbf{1}_{A_n^c}. \end{aligned} \quad (4.6)$$

Then, $\|m_n\|_\infty \leq 1$ implies that $m_n \mathbf{1}_{A_n^c} \rightarrow 0$ in L^2 . For the first term of m_n in (4.6) we have:

$$\begin{aligned} \int_{Q_{k+j}} |m_n \mathbf{1}_{A_n} - \tanh(\beta J * \hat{m})|^2 &\leq \int_{Q_{k+j} \cap A_n} |\tanh(\beta J * m_n - f(m_n)) - \tanh(\beta J * \hat{m})|^2 \\ &\leq c \int_{Q_{k+j} \cap A_n} |f(m_n)|^2 \rightarrow 0, \end{aligned}$$

since \tanh is uniformly Lipschitz continuous. Thus, $\lim_{n \rightarrow \infty} m_n = \tanh(\beta J * \hat{m})$ in $L^2(Q_{k+j})$. Therefore, since both $m_n \rightharpoonup \hat{m}$ in L_{loc}^2 and $m_n \rightarrow \tanh(\beta J * \hat{m})$ in Q_{k+j} we obtain that

$$\hat{m} = \tanh(\beta J * \hat{m}) \text{ in } Q_{k+j} \text{ and } f(\hat{m})(x) = 0 \ \forall x \in Q_{k+j}.$$

Now we obtain the contradiction. We have that

$$\inf_{\xi \in Q} \int_{Q_j} |m_n - \bar{m}_\xi|^2 \geq \vartheta, \quad \forall n,$$

which implies (since $\lim_{n \rightarrow \infty} m_n = \tanh(\beta J * \hat{m})$ in $L^2(Q_{k+j})$) that

$$\inf_{\xi \in Q} \int_{Q_j} |\tanh(\beta J * \hat{m}) - \bar{m}_\xi|^2 \geq \vartheta,$$

which (since $\hat{m} = \tanh(\beta J * \hat{m})$ in Q_{k+j}) in turn implies that $\hat{m} \in U_{Q,j,\vartheta}$. Furthermore, $\hat{m} \in V_{k,j}$ (closed in weak L^2). Thus, by lemma 4.3 there exists k^* such that $\int_{Q_{k+j}} |f(m)| > 0$ for all $m \in U_{Q,j,\vartheta}$. Contradiction, since this is not true for \hat{m} . \square

A similar result is true when the external conditions are in the plus or minus phase. Let

$$U_{Q,j,\vartheta}^{\pm} = \left\{ m \in L^{\infty}(\mathbb{R}, (-1, 1)) : Q \text{ is a } \pm \text{ contour for } m \text{ and } \int_{Q_j} |m \mp m_{\beta}|^2 \geq \vartheta \right\} \quad (4.7)$$

$$V_{k,j}^{\pm} = \left\{ m \in L^{\infty}(\mathbb{R}, (-1, 1)) : \eta^{(\zeta, \ell^{-})}(m; x) = \pm 1 \text{ for all } x \in B_{k,j}^{-} \cup B_{k,j}^{+} \right\}. \quad (4.8)$$

Then we also have the following:

Proposition 4.5. *For any $\vartheta > 0$, Q and Q_j there are k and $\rho > 0$ so that*

$$\inf_{m \in U_{Q,j,\vartheta}^{\pm} \cap V_{k,j}^{\pm}} \int_{Q_{k+j}} f(m)^2 \geq \rho. \quad (4.9)$$

With these ingredients we can conclude the proof of Theorem 4.1 following [11].

5. STRATEGY OF THE PROOF, GOOD AND BAD TIME INTERVALS

Given $\epsilon > 0$, we fix an orbit $\phi \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ as in Theorem 2.2 (neglecting from the notation the dependence on ϵ) and let $b(\phi)$ in (2.9) be the external force to which it corresponds. We decompose the time interval $[0, \epsilon^{-2}T]$ into subintervals $\{S[j, j+1), j \in \mathbb{N}\}$ of length $S > 0$. For $\kappa > 0$ we choose a parameter

$$\delta \equiv \delta(\epsilon) := |\log \epsilon|^{-\kappa} \quad (5.1)$$

and define

$$\phi^{(\delta, S)}(\phi; t) = \begin{cases} 1, & \text{if } \int_{jS}^{(j+1)S} \int_{\mathbb{R}} \mathcal{H}(\phi, \dot{\phi})(x, t) dx dt < \delta \\ 0, & \text{otherwise} \end{cases} \quad \text{for } t \in S[j, j+1). \quad (5.2)$$

To construct “time contours” we also define $\Phi^{(\delta, S)}(\phi; t)$ equal to 1 if $\phi^{(\delta, S)}(\phi; s) = 1$ for all $s \in S[j-1, j+1)$ and = 0 otherwise. We define $G_{\text{tot}} = \{t \leq \epsilon^{-2}T : \Phi^{(\delta, S)}(\phi; t) = 1\}$ and call t a “good time” and $S[j, j+1)$ a “good time interval” if they are contained in G_{tot} . Bad times and bad intervals are defined complementary.

Given the fact that it is too expensive to be away the instanton manifold (Corollary 4.2), the strategy now is to relate the cost functional to the cost of two mechanisms: translation of the interfaces and nucleation of new ones. The first can be achieved by relating the cost to the driving force of the motion of the interface and subsequently to its velocity. This is a valid approximation during the “good” time intervals. On the other hand, nucleations can only happen in the “bad” ones during which, the already existing interfaces cannot move too much because the overall cost is finite. We quantify all this in the next sections. We introduce the velocity of the formed interfaces and relate it to the cost. Contrary to [11], for the case of the cost derived via the large deviations this is not straightforward and new auxiliary profiles have to be introduced.

5.1. Parameters of the proof. We start by choosing some crucial parameters in the estimates. In Theorem 2.1 we saw that the cost of a nucleation (producing two fronts) is close to the cost of creating two interfaces, i.e., close to $2\mathcal{F}(\bar{m})$. Since the total cost is bounded by P , we obtain an upper bound (n^*) on the total number of fronts:

$$n^* = 1 + \frac{2P}{\mathcal{F}(\bar{m})}. \quad (5.3)$$

Moreover, following [11], for given $\gamma > 0$ we choose a critical value ℓ^* for the displacement of the fronts, after which we consider that a nucleation has occurred. This is determined to be such that the following holds:

$$|\mathcal{F}(\bar{m}_{(-\ell^*, \ell^*)}) - 2\mathcal{F}(\bar{m})| \leq \gamma, \quad \text{where} \quad \bar{m}_{(-\ell^*, \ell^*)} = \mathbf{1}_{x \geq 0} \bar{m}_{\ell^*} - \mathbf{1}_{x < 0} \bar{m}_{-\ell^*}. \quad (5.4)$$

This means that if the profile is made out of a combination of instantons whose centers are far enough (more than $2\ell^*$) then its free energy is well approximated by the number of such instantons times the cost of each one of them. Indeed, by the L^2 -continuity of $\mathcal{F}(\cdot)$, there is $\vartheta > 0$ so that for all m such that $d_{\mathcal{M}}(m) \leq \vartheta$ and with centers (ξ_1, \dots, ξ_n) , $n \leq n^*$, $\xi_{i+1} - \xi_i \geq 2\ell^*$, $\forall i$, we have that:

$$|\mathcal{F}(m) - n\mathcal{F}(\bar{m})| \leq \gamma. \quad (5.5)$$

However, it may happen that in a newly created nucleation the centers do not exceed the distance $2\ell^*$. These are called “incomplete nucleations” and we can neglect them arguing as in [11], [1] and [2] using the propositions below.

We first note that starting with such a profile, the free dynamics make it disappear within a finite time, depending on the distance ℓ (see [1], Proposition 7.1):

Proposition 5.1. *There is $\tau > 0$ so that for any positive $\ell \leq \ell^*$, the solution $v(x, s)$ of (2.1) starting from $\bar{m}_{(-\ell, \ell)}$ (as defined in (5.4)) verifies*

$$\sup_{x \in \mathbb{R}} |v(x, \tau) - m_\beta| \leq \vartheta.$$

This can be also used in a multi-instanton setting:

Proposition 5.2. *There is $L > 0$ for which the following holds. Let ℓ and τ be as in Proposition 5.1 and $\bar{\xi} = (\xi_1, \dots, \xi_n)$, $n \leq n^*$. Call \mathcal{I} the set of all even i such that $\xi_{i+1} - \xi_i \leq \ell$. Suppose \mathcal{I} non void and that for $j \notin \mathcal{I}$, $\xi_{j+1} - \xi_j \geq L$. Then the solution $w(x, t)$ of (2.1) which starts from $\bar{m}_{\bar{\xi}}$ is such that*

$$\sup_{x \in \mathbb{R}} |w(x, \tau) - \bar{m}_{\bar{\xi}^*}(x)| \leq \vartheta, \quad (5.6)$$

where $\bar{\xi}^*$ is obtained from $\bar{\xi}$ by dropping all pairs ξ_i, ξ_{i+1} , $i \in \mathcal{I}$.

Then, the same is true if we have an external force whose cost is controlled by a parameter $\alpha > 0$.

Proposition 5.3. *Let ℓ , τ , L , $\bar{\xi}$ and $\bar{\xi}^*$ as previously. Then there is $\alpha > 0$ such that if*

$$\|m - \bar{m}_{\bar{\xi}}\|_2 \leq \vartheta, \quad \int_0^\tau \int_{\mathbb{R}} |b(x, t)|^2 dx dt \leq \alpha, \quad (5.7)$$

then the solution $w(x, t)$ of (2.10) with force b and which starts from m is such that

$$\|w(x, \tau) - \bar{m}_{\bar{\xi}^*}(x)\|_2 \leq 4\vartheta. \quad (5.8)$$

From the previous propositions, we fix the parameters S and δ of our problem. Following the analysis in [11] we first choose the parameter S to be of order one such that:

$$S > 10^3 \max \left\{ \tau, \frac{3P}{c_{4.2}\rho}, \frac{4}{\omega} \right\}, \quad (5.9)$$

where ω is the spectral gap parameter given in Section 6. On the other hand, for δ a safe choice would be

$$\delta = 10^{-3} \min \left\{ \alpha, \frac{\vartheta}{c_{6.1}} \right\}, \quad \alpha \text{ and } c_{6.1} \text{ as in Proposition 5.3 and Proposition 6.1} \quad (5.10)$$

Hence, our choice in (5.1) satisfies the above criteria. With this choice of S and δ we have the following theorem:

Theorem 5.4. *Let ϕ satisfy (2.18) and let δ and S as above. Then:*

$$\text{number of bad time intervals} \leq \frac{2P}{\delta}. \quad (5.11)$$

If $S[j, j+1)$ is a good time interval, there is $t_1 \in S[j - \frac{1}{2}, j - \frac{1}{4})$ such that $d_{\mathcal{M}}(\phi(\cdot, t_1)) \leq \vartheta$.

Proof: Suppose that I is a bad interval and let I^- be its previous. Then inequality (5.2) cannot hold for both I and I^- since otherwise I would have been a good interval. Hence, the number of bad intervals is at most twice the number of intervals where (5.2) is not true. Thus,

$$P > \sum_{I: (5.2) \text{ is true}} + \sum_{I: (5.2) \text{ not true}} > \frac{1}{2}(\#\text{bad intervals})\delta$$

The second statement follows from Corollary 4.2. \square

5.2. Construction of auxiliary profiles ϕ_1 and m . Theorem 5.4 allows us to find times $t_j \in [j - \frac{1}{2}, j - \frac{1}{4}]S$, $j \in \mathcal{J} := \{1, 2, \dots, \frac{\epsilon^{-2}T}{S}\}$ for every good time interval $S[j, j+1)$, such that $d_{\mathcal{M}}(\phi(\cdot, t_j)) \leq \vartheta$. Then we define a new partition of $[0, \epsilon^{-2}T]$ as follows: if $S[j, j+1)$ is a good time interval in the original partition, we replace it by $[t_j, t_{j+1})$ and modify the neighbouring bad time intervals accordingly. For example, if the previous is bad, in the new partition it will be replaced by $[S(j-1), t_j)$. If $S[j+1, j+2)$ is a good time interval as well, then t_{j+1} are the ones given by Theorem 5.4, otherwise, $t_{j+1} := S(j+1)$. In this way, we obtain a new, slightly shifted, partition $\{[t_j, t_{j+1})\}_{j \in \mathcal{J}}$ of $[0, \epsilon^{-2}T]$. Note that in the new partition, the bad time intervals remain unchanged and this will be relevant in Section 8.

To prove Theorem 2.2, we want to derive lower bounds to the cost for a given profile given the condition on the total displacement. We estimate the cost of the given profile by assigning a notion of velocity to its fronts. The total displacement is then related to the motion of these fronts with the assigned velocity. We implement these during the good time intervals. Suppose t_j is the left endpoint of a maximal connected component G of G_{tot} . By the definition of t_j we have that $d_{\mathcal{M}}(\phi(\cdot, t_j)) \leq \vartheta$. For ϑ small enough, ϕ has only mixed contours which we denote by $\{\Gamma_i\}_{i=1}^k$, for some k odd. We call $\bar{\xi} = (\xi_1, \dots, \xi_k)$ its centers, ordered increasingly. In the first good time interval $[t_j, t_{j+1})$ of the connected component G , we construct an approximate (to ϕ) profile ϕ_1 as well as another orbit m as follows: First we truncate the forcing term $b(\phi)$. For $\lambda > 0$ we choose a threshold

$$\Delta \equiv \Delta(\epsilon) := |\log \epsilon|^{-\lambda}, \quad \lambda < \kappa, \quad (5.12)$$

for $\kappa > 0$ as in (5.1), and define a new external field

$$b_1(x, t) := b(\phi)(x, t) \mathbf{1}_{\{(x, t): |b(\phi)(x, t)| \leq \Delta(\epsilon)\}}. \quad (5.13)$$

Then we define the auxiliary profiles ϕ_1 and m to be the solutions of the following system:

$$\frac{d}{dt} \phi_1 = -\phi_1 + \tanh(\beta J * \phi_1) + \alpha b_1, \quad \phi_1(\cdot, t_j) = \phi(\cdot, t_j), \quad (5.14)$$

where

$$\alpha(x, t) := \left(\frac{1 - \bar{m}_{\bar{\xi}(t)}^2}{8} \right)^{1/2} \quad (5.15)$$

and $\bar{\xi}(t)$ are the approximate centers, defined in (6.2), of the profile m that satisfies the equation:

$$\frac{d}{dt} m = -m + \tanh(\beta J * m) + b(\phi_1), \quad m(\cdot, t_j) = m^{\text{in}}(\cdot). \quad (5.16)$$

Recall that the function b is given in (2.9). The initial condition $m^{\text{in}}(\cdot)$ is given below. Note that for the coefficient $\alpha(x, t)$ defined in (5.15) there exists a large constant $c_* > 0$ such that

$$\frac{1}{c_*} \leq \alpha(x, t) \leq 1, \quad \forall x, t. \quad (5.17)$$

Existence and uniqueness of solutions of the system (5.14)-(5.16) is proved in Appendix A. The idea for introducing the new force b_1 is that, following Appendix C, for forces of order $\Delta(\epsilon)$, the density \mathcal{H} of the cost is well approximated by b^2 . Moreover, an extra factor $\alpha(x, t)$ is needed in order to reconcile the coefficient of the asymptotics of \mathcal{H} (see (2.14)) with the space $L^2(\mathbb{R}, d\nu_{\bar{\xi}})$ in which we will be working later for the linearization around a moving instanton. Hence, the reason of introducing ϕ_1 is to have a profile whose centers are in a controlled distance from those of ϕ and additionally it has an external force which can be estimated by the cost. Then we use the idea in [11] of constructing sub-solutions (in our case of ϕ_1 rather than of ϕ) which start from an appropriately “regularized” initial profile and whose centers are ensured to move (being sub-solutions) at least as fast as the corresponding of ϕ . We denote this profile by m and note that, by a comparison theorem, it holds that

$m(x, t) \leq \phi_1(x, t)$ for $x \in \mathbb{R}$ and $t \in [t_j, t_{j+1})$. Next we present the initial condition $m^{\text{in}}(\cdot)$ by following the initialization procedure described in [11], Section 10.

5.3. Initial condition. Calling $\bar{\xi}(m) = (\xi_1(m), \dots, \xi_k(m))$ the centers of m at time t_j , we modify m in the following way:

Case 1: When $\xi_{j+1}(m) - \xi_j(m) > 2|\log \epsilon|^2$ for all j . We let $t_{\text{in}} = t_j$ and $m(\cdot, t_{\text{in}}^+) = m(\cdot, t_{\text{in}}^-)$.

Case 2: When $\xi_{j+1}(m) - \xi_j(m) \leq 2|\log \epsilon|^2$ for some j odd. We erase both centers for those j 's and we call the new configuration $\bar{\xi}^{(1)}(m)$, for which it holds that $\bar{m}_{\bar{\xi}^{(1)}(m)} \leq \bar{m}_{\bar{\xi}(m)}$. Then, we look at all even j in $\bar{\xi}^{(1)}(m)$ such that $2\ell^* \leq \xi_{j+1}(m) - \xi_j(m) \leq 2|\log \epsilon|^2$, ℓ^* as in Proposition 5.3 and we move each $\xi_j(m)$, $\xi_{j+1}(m)$ to $\xi'_j(m)$, $\xi'_{j+1}(m)$ so that

$$\xi_j(m) + \xi_{j+1}(m) = \xi'_j(m) + \xi'_{j+1}(m), \quad \xi'_{j+1}(m) - \xi'_j(m) = 2|\log \epsilon|^2.$$

We call $\bar{\xi}^{(2)}(m)$ the new configuration and $\bar{\xi}^{(3)}(m)$ the one obtained by $\bar{\xi}^{(2)}(m)$ following the same procedure as to obtain $\bar{\xi}^{(1)}(m)$. In $\bar{\xi}^{(3)}(m)$ the pairs $\xi_j(m)$, $\xi_{j+1}(m)$ with j even either satisfy $\xi_{j+1}(m) - \xi_j(m) \geq 2|\log \epsilon|^2$ or $\xi_{j+1}(m) - \xi_j(m) \leq 2\ell^*$. Case 2 is when $\xi_{j+1}(m) - \xi_j(m) \geq 2|\log \epsilon|^2$ for all j . Then, we define

$$\tilde{m}(x, t_j) = \min\{m(x, t_j), \bar{m}_{\bar{\xi}^{(3)}(m)}\}$$

$t_{\text{in}} = t_j$ and $m(\cdot, t_{\text{in}}^+) = m(\cdot, t_j)$.

Case 3: This case covers all remaining possibilities in the previous case when in $\bar{\xi}^{(3)}(m)$ there is at least a pair $\xi_j(m)$, $\xi_{j+1}(m)$ with j even satisfying $\xi_{j+1}(m) - \xi_j(m) \leq 2\ell^*$. In that case, we let $t_{\text{in}} = t_j + \tau$, τ as in Proposition 5.3 and $m(\cdot, t_{\text{in}}^+)$ is the solution at time $t_j + \tau$ of (2.1) starting from $\tilde{m}(x, t_j)$. We finally define $m^{\text{in}}(\cdot) := m(\cdot, t_{\text{in}})$.

As a result of this initialization procedure, we have that for all $\epsilon > 0$ small enough, the centers of $m(\cdot, t_j)$ have mutual distance $\geq |\log \epsilon|^2$ and $d_{\mathcal{M}}(\phi(\cdot, t_j)) \leq 6\vartheta$. To prove this, we use Proposition 5.3 with external force $b := b(\phi_1) = \alpha b_1$. In such a case, we have that $\int b^2$ is related to the cost since we apply it within a good time interval; hence the requirement (5.7) is satisfied. In the next section we show that in the good time interval $[t_j, t_{j+1})$ the solution $m(t, \cdot)$ of (5.16) follows closely a moving instanton $\bar{m}_{\bar{\xi}(t)}$, where $\bar{\xi}(t)$ are the centers of $m(t, \cdot)$.

6. LINEARIZATION AROUND A MOVING INSTANTON

By the constuction in the previous section, we have that in the good time interval $[t_j, t_{j+1})$ the profile m solves the equation

$$\frac{d}{dt}m = -m + \tanh(\beta J * m) + b(\phi_1), \quad m(\cdot, t_j) = m^{\text{in}}(\cdot), \quad (6.1)$$

where the initial condition $m^{\text{in}}(\cdot)$ is given by the same initialization as in [11], i.e., it has an odd number k of mixed contours at mutual distance $\geq |\log \epsilon|^2$; moreover $d_{\mathcal{M}}(m^{\text{in}}(\cdot)) \leq 6\vartheta$.

Choice of parameters. In the sequel, $\omega > 0$ is the “spectral gap parameter” defined in [14], c is given in (6.10) and $\epsilon_1 < \frac{\omega}{8c}$. Calling $\bar{\xi}(t) = (\xi_1(t), \dots, \xi_k(t))$ the centers of $m(\cdot, t)$,

$t \geq t_j$, we define the approximate centers $\tilde{\xi}(t) = (\tilde{\xi}_1(t), \dots, \tilde{\xi}_k(t))$ and the deviation $u(\cdot, t)$ as follows:

$$(\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\tilde{\xi}_i(t)}, [m(\cdot, t) - \sigma_i \bar{m}_{\tilde{\xi}_i(t)}])_{L^2(d\nu)} = 0, \quad u(\cdot, t) = m(\cdot, t) - \bar{m}_{\tilde{\xi}(t)}, \quad (6.2)$$

where

$$A_{\alpha^*} := \left\{ x \in \mathbb{R} : \int_{t_{j-1}}^{t_{j+1}} b_1^2(x, s) ds \leq \alpha^* \right\} \quad (6.3)$$

for α^* small enough and $\sigma_i = 1$ [$\sigma_i = -1$] if i is odd [even] and $\tilde{\xi}_i(t)$ in the i -th mixed contour of $m(\cdot, t)$. From the definition of A_{α^*} we also have that

$$|A_{\alpha^*}^c| \leq \frac{8}{\alpha^*} \int_{t_{j-1}}^{t_{j+1}} \|\alpha b_1(s)\|_{L^2(d\nu)}^2 ds, \quad (6.4)$$

where

$$d\nu(x) := \frac{1}{1 - \bar{m}_{\tilde{\xi}(t)}^2} dx.$$

Moreover, we call $\Lambda_i(t)$, $i = 1, \dots, k$, the open intervals $\frac{1}{2}(\tilde{\xi}_{i-1}(t) + \tilde{\xi}_i(t), \tilde{\xi}_{i+1}(t) + \tilde{\xi}_i(t))$, with $\tilde{\xi}_0(t) = -\infty$ and $\tilde{\xi}_{k+1}(t) = +\infty$. We have the following estimate

$$|\tilde{\xi}_i(t) - \xi_i(t)| + \|u(\cdot, t) - \{m(\cdot, t) - \bar{m}_{\tilde{\xi}(t)}\}\|_{L^2(d\nu)} \leq \frac{c}{\alpha^*} \int_{t_{j-1}}^{t_{j+1}} \|\alpha b_1(s)\|_{L^2(d\nu)}^2 ds. \quad (6.5)$$

In the next proposition we give upper bounds for displacements of centers with i odd and lower bounds for those with i even. In the proof, we follow the strategy in [11] with the exception of having a different operator and therefore we have to work in a appropriately weighted space.

Proposition 6.1. *There is a constant $c_{6.1} > 0$, so that for ϑ and δ small enough and for all $t \in [t_j, t_{j+1}]$, we have the following bounds:*

$$\|u(\cdot, t)\|_{L^2(d\nu)}^2 \leq e^{-(t-t_j)\omega} \|u(\cdot, t_{\text{in}})\|_{L^2(d\nu)}^2 + c_{6.1} S U_j^2, \quad (6.6)$$

$$\sigma_i [\xi_i(t) - \xi_i(t_{\text{in}})] \leq -\frac{1}{\|\bar{m}'\|_2^2} \int_{t_j}^t (\alpha b_1, \bar{m}'_{\tilde{\xi}_i(t)})_{L^2(d\nu)} + c_{6.1} [\|u(\cdot, t_j)\|_{L^2(d\nu)}^2 + U_j^2], \quad (6.7)$$

where $i = 1, \dots, k$ and

$$U_j^2 = \int_{t_j}^t \|\alpha b_1\|_{L^2(d\nu)}^2 + S R_{\max}, \quad R_{\max} = c_{6.1} e^{-\alpha |\log \epsilon|^2/2}. \quad (6.8)$$

Note that $R_{\max} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proof. Let

$$L : L^2(\mathbb{R}, d\nu) \rightarrow L^2(\mathbb{R}, d\nu), \quad (Lu)(x) := -u(x) + (1 - \bar{m}_{\tilde{\xi}(t)}^2)(\beta J * u)(x),$$

where

$$d\nu(x) := \frac{dx}{1 - \bar{m}_{\tilde{\xi}(t)}^2(x)}.$$

For $x \in \Lambda_i$, we have

$$\frac{du(x, t)}{dt} = \sigma_i \dot{\tilde{\xi}}_i(t) \bar{m}'_{\tilde{\xi}_i(t)} + Lu(x, t) + \tilde{R}(u) + \alpha b_1(x, t), \quad (6.9)$$

where

$$\tilde{R}(u) := G''(\beta J * (\bar{m}_{\tilde{\xi}(t)} + (1 - \mu_0)\lambda_0 u))(\beta J * u)^2,$$

with

$$0 \leq \lambda_0, \mu_0 \leq 1$$

and

$$G(x) := \tanh x.$$

It is an easy calculation to show that

$$\|\tilde{R}(u)\|_{L^1(d\nu)} \leq c\|u\|_{L^2(d\nu)}^2. \quad (6.10)$$

By multiplying (6.9) by $u(\cdot, t)\mathbf{1}_{A_{\alpha^*}}$ and integrating over space we obtain:

$$\frac{d}{dt} \left(\frac{1}{2} \|u\mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)}^2 \right) = (u\mathbf{1}_{A_{\alpha^*}}, Lu)_{L^2(d\nu)} + (u\mathbf{1}_{A_{\alpha^*}}, \tilde{R}(u))_{L^2(d\nu)} + \int_{\mathbb{R}} u\mathbf{1}_{A_{\alpha^*}} \alpha b_1 d\nu + R(t), \quad (6.11)$$

where

$$R(t) = \sum_{i=1}^k \sigma_i \dot{\tilde{\xi}}_i(t) \left(\int_{\Lambda_i} \bar{m}_{\tilde{\xi}_i(t)} u\mathbf{1}_{A_{\alpha^*}} d\nu + \int_{\Lambda_i} \bar{m}_{\tilde{\xi}_i(t)} \frac{\bar{m}_{\tilde{\xi}_i(t)}}{1 - \bar{m}_{\tilde{\xi}_i(t)}^2} u^2 \mathbf{1}_{A_{\alpha^*}} d\nu \right). \quad (6.12)$$

By (6.4),

$$\left| (u\mathbf{1}_{A_{\alpha^*}}, Lu)_{L^2(d\nu)} - (u\mathbf{1}_{A_{\alpha^*}}, L(u\mathbf{1}_{A_{\alpha^*}}))_{L^2(d\nu)} \right| \leq \frac{32}{\alpha^*} \int_{t_{j-1}}^{t_{j+1}} \|\alpha b_1(s)\|_{L^2(d\nu)}^2 ds.$$

By the spectral gap property, $(u\mathbf{1}_{A_{\alpha^*}}, Lu\mathbf{1}_{A_{\alpha^*}})_{L^2(d\nu)} \leq -\omega\|u\|_{L^2(d\nu)}$ and by using a similar estimate on $\|u\|_{L^\infty}$ as in Theorem C.3 of Appendix in [11] in order to bound the second term in (6.11), we obtain:

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|u\|_{L^2(d\nu)}^2 \right) &\leq -\omega\|u\mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)} + c(\epsilon_1 + c_1\|u\|_{L^2(d\nu)})^{2/3}\|u\mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)} \\ &\quad + (u\mathbf{1}_{A_{\alpha^*}}, \alpha b_1)_{L^2(d\nu)} + c' \int_{t_j}^{t_{j+1}} \|\alpha b_1(s)\|_{L^2(d\nu)}^2 ds + R(t). \end{aligned}$$

Let

$$\tau := \inf \left\{ t : \|u(\cdot, t)\|_{L^2(d\nu)}^{2/3} > \frac{\omega}{8cc_1} \right\}. \quad (6.13)$$

Bounding $\|(u\mathbf{1}_{A_{\alpha^*}}, \alpha b_1)\|_{L^2(d\nu)} \leq \frac{2\|\alpha b_1\|_{L^2(d\nu)}^2}{\omega} + \frac{\omega\|u\mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)}^2}{4}$, for all times $t \in [t_j, t_{j+1}]$ such that $t < \tau$ we have:

$$\frac{d}{dt} \left(\frac{1}{2} \|u\mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)}^2 \right) \leq -\frac{\omega}{2} \|u\mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)}^2 + \frac{2}{\omega} \|\alpha b_1\|_{L^2(d\nu)}^2 + R(t),$$

i.e., for $t^* = \min\{\tau, t_{j+1}\}$ we obtain

$$\|\mathbf{1}_{A_{\alpha^*}} u(\cdot, t^*)\|_{L^2(d\nu)}^2 \leq e^{-(t^*-t_j)\omega} \|u(\cdot, t_j)\|_{L^2(d\nu)}^2 + c_{6.1} \left(\int_{t_j}^{t^*} \|\alpha b_1(s)\|_{L^2(d\nu)}^2 + SR_{\max} \right),$$

with R_{\max} defined in (6.8). Since

$$\|u\|_{L^2(d\nu)}^2 \leq \|\mathbf{1}_{A_{\alpha^*}} u\|_{L^2(d\nu)}^2 + \frac{4}{\alpha^*} \int_{t_j}^{t_{j+1}} \|\alpha b_1(s)\|_{L^2(d\nu)}^2,$$

we have

$$\|u(\cdot, t^*)\|_{L^2(d\nu)}^2 \leq e^{-(t^*-t_j)\omega} \|u(\cdot, t_j)\|_{L^2(d\nu)}^2 + c_{6.1} \left(\int_{t_j}^{t^*} \|\alpha b_1(s)\|_{L^2(d\nu)}^2 + SR_{\max} \right).$$

By the choice of δ in (5.10) and (C.1) we have

$$c_{6.1} \int_{t_j}^{t^*} \|\alpha b_1(s)\|_{L^2(d\nu)}^2 + SR_{\max} \leq c_{6.1} \left(\frac{1}{1 - c_*^2 C \Delta(\epsilon)} \delta + SR_{\max} \right) \leq 10^{-3}.$$

Thus, for δ, ϑ and ϵ small enough, $\|u(\cdot, t^*)\|_{L^2(d\nu)}^2 \leq (\frac{\omega}{8cc_1})^3$ and hence $t^* = t_{j+1}$.

For the proof of (6.7), we multiply (6.9) by $\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\tilde{\xi}_i(t)}$ and estimate $(\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\tilde{\xi}_i(t)}, u_t)_{L^2(d\nu)}$ by first writing (6.2) as

$$(\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\tilde{\xi}_i(t)}, \sigma_i \bar{m}_{\tilde{\xi}_i(t)} - \bar{m}_{\tilde{\xi}_i(t)})_{L^2(d\nu)} = (\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\tilde{\xi}_i(t)}, u)_{L^2(d\nu)}, \quad (6.14)$$

after adding and subtracting $\bar{m}_{\tilde{\xi}_i(t)}$. Since the measure $d\nu$ depends on time, we also have:

$$\begin{aligned} \frac{d}{dt} (\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\tilde{\xi}_i(t)}, u)_{L^2(d\nu)} &= (\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\tilde{\xi}_i(t)}, u_t)_{L^2(d\nu)} + (\mathbf{1}_{A_{\alpha^*}} \bar{m}''_{\tilde{\xi}_i} \sigma_i \dot{\tilde{\xi}}_i, u)_{L^2(d\nu)} \\ &\quad + \sum_j \int_{\Lambda_j} \bar{m}'_{\tilde{\xi}_i} \mathbf{1}_{A_{\alpha^*}} u \frac{2\bar{m}_{\tilde{\xi}_j} \bar{m}'_{\tilde{\xi}_j} \dot{\tilde{\xi}}_j}{(1 - \bar{m}_{\tilde{\xi}_j}^2)^2} dx. \end{aligned} \quad (6.15)$$

We obtain:

$$\begin{aligned}
(\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\xi_i}, u_t)_{L^2(d\nu)} &= \dot{\xi}_i \left\{ (\mathbf{1}_{A_{\alpha^*}} \bar{m}''_{\xi_i}, u)_{L^2(d\nu)} + (\mathbf{1}_{A_{\alpha^*}} \bar{m}''_{\xi_i}, \bar{m}_{\xi} - \sigma_i \bar{m}_{\xi_i})_{L^2(d\nu)} \right\} \\
&\quad - \sum_{j \neq i} \left(\mathbf{1}_{A_{\alpha^*}} \mathbf{1}_{\Lambda_j} \bar{m}'_{\xi_i}, (\sigma_i \dot{\xi}_i \bar{m}'_{\xi_i} - \sigma_j \dot{\xi}_j \bar{m}'_{\xi_j}) \right)_{L^2(d\nu)} \\
&\quad + \sum_{j \neq i} \int_{\Lambda_j} 2 \mathbf{1}_{A_{\alpha^*}} u \bar{m}_{\xi_j} \frac{\bar{m}'_{\xi_i} \bar{m}'_{\xi_j}}{1 - \bar{m}_{\xi_j}^2} d\nu \\
&\quad - \sum_{j \neq i} \int_{\Lambda_j} \mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\xi_i} (\sigma_i \bar{m}_{\xi_i} - \bar{m}_{\xi}) \frac{1}{1 - \bar{m}_{\xi_j}^2} 2 \bar{m}_{\xi_j} \bar{m}'_{\xi_j} \sigma_j \dot{\xi}_j d\nu. \quad (6.16)
\end{aligned}$$

On the other hand, in (6.9) we have:

$$(\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\xi_i}, Lu)_{L^2(d\nu)} = (u, L \bar{m}'_{\xi_i})_{L^2(d\nu)}, \quad \text{with } |L \bar{m}'_{\xi_i}| \leq R_{\max}.$$

Thus, from (6.9) and (6.16) we obtain:

$$\begin{aligned}
&\sigma_i \dot{\xi}_i \left[\|\bar{m}'_{\xi_i} \mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)}^2 - \sigma_i \{ (\mathbf{1}_{A_{\alpha^*}} \bar{m}''_{\xi_i}, u)_{L^2(d\nu)} + (\mathbf{1}_{A_{\alpha^*}} \bar{m}''_{\xi_i}, \bar{m}_{\xi} - \sigma_i \bar{m}_{\xi_i})_{L^2(d\nu)} \} \right] \\
&+ \sum_{j \neq i} \left(\mathbf{1}_{A_{\alpha^*}} \mathbf{1}_{\Lambda_j} \bar{m}'_{\xi_i}, (\sigma_i \dot{\xi}_i \bar{m}'_{\xi_i} - \sigma_j \dot{\xi}_j \bar{m}'_{\xi_j}) \right)_{L^2(d\nu)} - \sum_{j \neq i} \int_{\Lambda_j} \mathbf{1}_{A_{\alpha^*}} 2u \bar{m}_{\xi_j} \frac{\bar{m}'_{\xi_i} \bar{m}'_{\xi_j}}{1 - \bar{m}_{\xi_j}^2} d\nu \\
&\quad + \sum_{j \neq i} \int_{\Lambda_j} \mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\xi_i} (\sigma_i \bar{m}_{\xi_i} - \bar{m}_{\xi}) \frac{1}{1 - \bar{m}_{\xi_j}^2} 2 \bar{m}_{\xi_j} \bar{m}'_{\xi_j} \sigma_j \dot{\xi}_j d\nu \\
&\leq -(\bar{m}'_{\xi_i}, \alpha b_1)_{L^2(d\nu)} + |A_{\alpha^*}^c| + c' c \|\mathbf{1}_{A_{\alpha^*}} u\|_{L^2(d\nu)}^2 + R_{\max}
\end{aligned}$$

which has the form:

$$\sigma_i \|\bar{m}'\|_{L^2(d\nu)}^2 \dot{\xi}_i(t) \leq \beta_i + \sum_{j=1}^k a_{i,j} |\dot{\xi}_j|, \quad (6.17)$$

where

$$\beta_i = (\mathbf{1}_{A_{\alpha^*}} \bar{m}''_{\xi_i}, u)_{L^2(d\nu)} - (\bar{m}'_{\xi_i}, \alpha b_1)_{L^2(d\nu)} + c' c \|u\|_{L^2(d\nu)}^2 + |A_{\alpha^*}^c| + R_{\max}, \quad (6.18)$$

with

$$\begin{aligned}
|\beta_i + (\bar{m}'_{\xi_i}, \alpha b_1)_{L^2(d\nu)}| &\leq c'' [e^{-(t-t_{\text{in}})\omega} \|u(\cdot, t_{\text{in}})\|_{L^2(d\nu)}^2 \\
&\quad + \int_{t_{\text{in}}}^t \|\alpha b_1\|_{L^2(d\nu)}^2 ds + S R_{\max} + \|1 - \mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)} \|\alpha b_1\|_{L^2(d\nu)}]
\end{aligned}$$

and

$$\begin{aligned}
a_{i,j} &= (\mathbf{1}_{\Lambda_j} \bar{m}''_{\xi_j}, \bar{m}_{\xi_i} - \sigma_j \bar{m}_{\xi_j}) + (\mathbf{1}_{\Lambda_j} \bar{m}'_{\xi_i}, \bar{m}'_{\xi_j})_{L^2(d\nu)} \\
&\quad + \int_{\Lambda_j} 2u \bar{m}_{\xi_j} \frac{\bar{m}'_{\xi_i} \bar{m}'_{\xi_j}}{1 - \bar{m}_{\xi_j}^2} d\nu - \int_{\Lambda_j} \bar{m}'_{\xi_i} (\sigma_i \bar{m}_{\xi_i} - \bar{m}_{\xi}) \frac{2 \bar{m}_{\xi_j} \bar{m}'_{\xi_j}}{1 - \bar{m}_{\xi_j}^2} d\nu. \quad (6.19)
\end{aligned}$$

Then we conclude the proof in the same fashion as in [11] by estimating $a_{i,j}$, since ξ_i and ξ_j are well separated. \square

Concluding this section, we recall that we constructed $m(t, \cdot)$ for $t \in [t_j, t_{j+1}]$ and obtained estimates for the error $\|m(\cdot, t) - \bar{m}_{\xi(t)}\|_{L^2(d\nu)}^2$. Next we define $m(\cdot, t_{j+1}^+)$ in order to apply this linearization procedure in the whole of the maximal connected component G .

7. FROM A GOOD TIME INTERVAL TO THE NEXT

The result of Proposition 6.1 ensures that during the good time interval $[t_j, t_{j+1})$ the solution of (6.1) is close to a moving instanton. More precisely, by (5.10) we have that $c_{6.1}U_j^2 \leq \vartheta$ and by (5.9) that $e^{-\omega S} \leq 1/2$. Then by (6.6) we get, supposing ϵ small enough,

$$\|u(\cdot, t_{j+1})\|_{L^2(d\nu)}^2 \leq e^{-\omega S} \|u(t_j)\|_{L^2(d\nu)}^2 + c_{6.1}U_j^2 \leq 4\vartheta. \quad (7.1)$$

Furthermore, since $\xi_{i+1}(t_{j+1}) - \xi_i(t_{j+1}) \geq |\log \epsilon|^2/2$, as we have seen in the course of the proof of Proposition 6.1, it follows from (3.13) that for ϵ small enough,

$$d_{\mathcal{M}}(m(\cdot, t_{j+1})) \leq 5\vartheta. \quad (7.2)$$

We introduce the notion of *velocity of a front* $\bar{m}_{\xi_i}(t)$, by defining:

$$v_i^0(t) := \sigma_i \frac{1}{\|\bar{m}'\|_{L^2(d\nu)}^2} \left| (\alpha b_1, \bar{m}'_{\xi_i(t)})_{L^2(d\nu)} \right|, \quad (7.3)$$

where again $\sigma_i = 1$ [$\sigma_i = -1$] if i is odd [even]. Moreover, we want to control the position of the centers of $m(\cdot, t)$, so we denote by $r_i(t)$ the leftmost [rightmost] position of the center ξ_i of $m(\cdot, t)$, for i odd [even], taking into account the error in determining the position ξ_i . Thus, the position $r_i(t)$ will be given by ξ_i plus the integral of the velocity induced by the error $\|m(\cdot, t) - \bar{m}_{\xi(t)}\|_{L^2(d\nu)}^2$. We define:

$$v_i(t) := v_i^0(t) + \sigma_i c_{6.1} \left(U_j^2 + \|u(\cdot, t_j)\|_{L^2(d\nu)}^2 \right), \quad (7.4)$$

$$r_i(t) := \xi_i(t_j) + \int_{t_j}^t v_i(s), \quad \bar{r}(t) = (r_1(t), \dots, r_k(t)). \quad (7.5)$$

Notice that $\bar{r}(t) \leq \bar{\xi}(t)$ for $t \in [t_j, t_{j+1})$, where the partial order is defined as:

$$(\xi_1, \dots, \xi_k) \geq (\xi'_1, \dots, \xi'_{k'}) \Leftrightarrow \bar{m}_{(\xi_1, \dots, \xi_k)} \geq \bar{m}_{(\xi'_1, \dots, \xi'_{k'})}. \quad (7.6)$$

In particular, if $k = k'$,

$$(\xi_1, \dots, \xi_k) \geq (\xi'_1, \dots, \xi'_k) \Leftrightarrow \xi_i \leq \xi'_i, i \text{ odd}, \quad \xi_i \geq \xi'_i, i \text{ even}. \quad (7.7)$$

By the definition of t_{j+1} we know that $d_{\mathcal{M}}(\phi(\cdot, t_{j+1})) \leq \vartheta$. Suppose now that, for $\epsilon > 0$ small enough, $\phi(\cdot, t_{j+1})$ has k' -many mixed contours $\{\tilde{\Gamma}_i\}_{i=1, \dots, k'}$, k' odd, with $\|\mathbf{1}_{\tilde{\Gamma}_i}(\phi - \bar{m}_{\tilde{\xi}_i})\|_{L^2} \leq \vartheta$ for some $\tilde{\xi}_i \in \tilde{\Gamma}_i$, $i = 1, \dots, k'$. Note that in general $k' \neq k$ (since m has been re-initialized at t_j and some fronts might have been cancelled). Then by Theorem 3.3 we have that there

exist unique centers $\{\xi_i(\phi)(t_{j+1})\}_{i=1,\dots,k}$ of $\phi(\cdot, t_{j+1})$. The strategy goes as follows: note that since (using (5.17))

$$|b(\phi_1)| = |\alpha b_1| = \left| \left(\frac{1 - \bar{m}_{\xi(t)}^2}{8} \right)^{1/2} b_1 \right| \leq |b_1| \leq |b(\phi)|,$$

the profile $\phi_1(t_{j+1})$ is expected to have its odd [even] indexed centers on the left [right] of the corresponding centers of $\phi(t_{j+1})$. On the other hand, the profile $m(t_{j+1})$, being a sub-solution of the equation $b(m) = b(\phi_1)$, with initial condition $m(t_j)$ re-initialized as before, it has its odd [even] centers on the right [left] of the corresponding centers of $\phi_1(t_{j+1})$. However, it is not guaranteed that this is also the case with the centers of $\phi(t_{j+1})$. Therefore, since in the next good time interval we choose $\phi_1(\cdot, t_{j+1}^+) := \phi(\cdot, t_{j+1}^+)$ we need to re-initialize $m(\cdot, t_{j+1}^+)$ to be such that $m(\cdot, t_{j+1}^+) \leq \phi_1(\cdot, t_{j+1}^+)$ and keep track of the relevant error. As a result of the initialization, the profile $m(t_{j+1})$ may have fewer centers than $\phi_1(\cdot, t_{j+1})$.

We estimate the distance between the corresponding centers of ϕ and m at t_{j+1} , when both are close to the manifold \mathcal{M} . Recall also that, by the initialization, the centers at t_j^+ have mutual distance $\geq |\log \epsilon|^2$. To perform our estimate we introduce an auxiliary profile ϕ_2 by putting as forcing term only b_1 with the same initial condition. For $t \in [t_j, t_{j+1})$ we have:

$$\|\phi(t) - \phi_2(t)\|_{L^1} \leq \int_{t_j}^t e^{-(t-s+t_j)} \beta \|J\|_{L^1} \|\phi(s) - \phi_2(s)\|_{L^1} ds + \int_{t_j}^t \int_{\mathbb{R}} e^{-(t-s+t_j)} |b - b_1| dx ds,$$

where

$$\int_{t_j}^t \int_{\mathbb{R}} e^{-(t-s+t_j)} |b - b_1| dx ds \leq \int_{|b| > \Delta(\epsilon)} |b| dx ds.$$

In the good time interval $[t_j, t_{j+1})$ we define the quantity:

$$\delta_j := \int_{t_j}^{t_{j+1}} \int_{\mathbb{R}} \mathcal{H}(b, u, w)(x, s) ds dx, \quad (7.8)$$

in which case it is of the order $\delta(\epsilon)$. From (2.14) we obtain that:

$$\begin{aligned} \delta_j = \int_{t_j}^{t_{j+1}} \int_{\mathbb{R}} \mathcal{H}(b, u, w)(x, s) ds dx &\geq \int_{\{|b| > \Delta(\epsilon)\}} \mathcal{H}(b, u, w)(x, s) ds dx \\ &\geq C \int_{\{|b| > \Delta(\epsilon)\}} |b| \log(|b| + 1) ds dx \\ &\geq C \int_{\{|b| > \Delta(\epsilon)\}} |b| \log(1 + \Delta(\epsilon)) ds dx. \end{aligned}$$

Thus, (since $\|J\|_{L^1} = 1$)

$$\|\phi(\cdot, t) - \phi_2(\cdot, t)\|_{L^1} \leq \beta \int_{t_j}^t e^{-(t-s+t_j)} \|\phi(\cdot, s) - \phi_2(\cdot, s)\|_{L^1} ds + \frac{\delta_j}{C \log(1 + \Delta(\epsilon))} \quad (7.9)$$

and for a new constant $C > 0$ by Gronwall's lemma we obtain that

$$\|\phi(\cdot, t_{j+1}) - \phi_2(\cdot, t_{j+1})\|_{L^1} \leq Ce^{(2+\beta)S} \frac{\delta_j}{\Delta(\epsilon)}. \quad (7.10)$$

On the other hand, comparing to m we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} (\phi_2 - m)^2(x, t) dx = \\ &= -2 \int_{\mathbb{R}} (\phi_2 - m)^2(x, t) dx + 2 \int_{\mathbb{R}} (1 - \alpha) b_1(x, t) (\phi_2 - m)(x, t) dx \\ & \quad + 2 \int_{\mathbb{R}} (\phi_2 - m)(x, t) (\tanh(\beta J * \phi_2(x, t)) - \tanh(\beta J * m^0(x, t))) dx \\ &\leq C \int_{\mathbb{R}} (\phi_2 - m)^2(x, t) dx + c \int_{\mathbb{R}} (1 - \alpha)^2 b_1^2(x, t) dx. \end{aligned}$$

Since from (5.17) it holds that $1 - \alpha \leq (c^* - 1)\alpha$, applying Gronwall's inequality and using (C.1) we obtain

$$\begin{aligned} \|\phi_2(\cdot, t) - m(\cdot, t)\|_{L^2}^2 &\leq ce^{(2+\beta)(t-t_j)} \int_{\mathbb{R}} \int_{t_j}^t \alpha^2 b_1^2 ds dx \\ &\leq ce^{(2+\beta)S} \frac{1}{1 - c_*^2 C \Delta(\epsilon)} \delta_j, \end{aligned} \quad (7.11)$$

for ϵ small enough so that $c_*^2 C \Delta(\epsilon) < 1$. Thus, since $\|\phi(\cdot, t) - \phi_2(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq 2$, (7.9) and (7.11) yield

$$\begin{aligned} \|\phi(\cdot, t) - m(\cdot, t)\|_{L^2(\mathbb{R})}^2 &\leq 2\|\phi_2(\cdot, t) - m(\cdot, t)\|_{L^2(\mathbb{R})}^2(t) + 4\|\phi(\cdot, t) - \phi_2(\cdot, t)\|_{L^1(\mathbb{R})} \\ &\leq C \frac{\delta_j}{\Delta(\epsilon)} + ce^{(2+\beta)S} \frac{1}{1 - c_*^2 C \Delta(\epsilon)} \delta_j =: S_\epsilon^j, \end{aligned} \quad (7.12)$$

where by choosing $\kappa < \lambda$ in the definition of $\Delta(\epsilon)$ in (5.12), we have that $S_\epsilon^j \rightarrow 0$ as $\epsilon \rightarrow 0$. Using the above estimate and the fact that both m and ϕ are close to the manifold at time t_j , we obtain that

$$|\xi(\phi)(t_{j+1}) - \xi(m)(t_{j+1})| \leq \|\bar{m}_{\bar{\xi}(m)} - \bar{m}_{\bar{\xi}(\phi)}\| \leq S_\epsilon^j + 6\vartheta. \quad (7.13)$$

Next, recalling the definition of $\bar{r}(t)$ in (7.5), in order to define $r_i(t_{j+1}^+)$ we consider the quantity

$$\hat{r}_i(t_{j+1}) := r_i(t_{j+1}) + \sigma_i S_\epsilon^j \quad (7.14)$$

and we erase all pairs $i, i+1$ such that $\hat{r}_{i+1}(t_{j+1}) - \hat{r}_i(t_{j+1}) \leq |\log \epsilon|^2$. Then we let

$$r_i(t_{j+1}^+) := \hat{r}_i(t_{j+1}),$$

if no such erasing has occurred for the index i . Otherwise, we let $r_i(t_{j+1}^+) := \emptyset$.

In Section 9 we introduce the notion of particles while referring to the fronts and we say that in this case the particles i and $i+1$ have collided and, due to this collision, they disappeared.

We will also write that $r_i(t) = r_{i+1}(t) = \emptyset$ for $t > t_{j+1}$. Moreover, note that the function $\bar{r}(t)$ has jumps at the times between good time intervals and this fact will be taken into account in the estimation of the total displacement and the corresponding “macroscopic” cost expressed in terms of the cost due to the motion of the particles. For the re-initialization at t_{j+1}^+ we define:

$$m(\cdot, t_{j+1}^+) := \min\{\phi(\cdot, t_{j+1}), \bar{m}_{r_i(t_{j+1}^+)}(\cdot)\}. \quad (7.15)$$

In this way we ensure that $m(\cdot, t_{j+1}^+) \leq \phi(\cdot, t_{j+1})$ as well as that $r_i(t_{j+1}^+)$ is a lower [upper] bound of $\xi_i(m(\cdot, t_{j+1}^+))$ for i odd [even]. Thus, taking ϵ small enough we have that $d_{\mathcal{M}}(m(\cdot, t_{j+1}^+)) \leq 20\vartheta$ and that its centers have mutual distance $\geq |\log \epsilon|^2$. So we can repeat the same procedure for the next good time interval $[t_{j+1}, t_{j+2})$.

8. DISPLACEMENT DURING THE BAD TIME INTERVALS

From (5.11) the maximal length of the connected component of bad time intervals is bounded by $S \frac{P}{\delta(\epsilon)} \ll |\log \epsilon|^2$ for the choice of $\delta(\epsilon)$ made in (5.12). Moreover, the applied force b can be related to and bounded by the cost. Therefore, the displacement of the already existing centers should be smaller than $|\log \epsilon|^2$, which is the distance between the appropriately initialized centers of the interfaces. Similarly, the newly nucleated fronts are also at a distance from each other smaller than $|\log \epsilon|^2$ even at the end of the connected component of the bad time intervals. Hence, overall the motion during the bad time intervals will be negligible macroscopically.

Suppose that $[t_{j'}, t_{j''})$ is a connected component of bad time intervals. Recalling the construction of the partition of good and bad time intervals in subsection 5.2, we have that $t_k = kS$, for all $j' \leq k \leq j''$, $k \in \mathbb{N}$. In the connected component of bad time intervals we define the profile m by solving the equation

$$\frac{d}{dt}m = -m + \tanh(\beta J * m) + b(\phi), \quad (8.1)$$

with initial condition the profile $m(t_{j'}, \cdot)$ as we obtained it from the previous good time interval. Invoking again Corollary 4.2 and the choice of S for the profile m constructed above, for $j' + 1 \leq k \leq j''$ there exist $\bar{t}_k \in [t_j, t_{j+1})$ with $m(\bar{t}_k, \cdot)$ close to \mathcal{M} .

We compare the solution m to the solution m^0 of the same equation without the forcing term $b(\phi)$ for the interval $[t_{j'}, \bar{t}_{j'+1})$, both with the same initial condition. To do that we compare both of them to the auxiliary profile ϕ_2 generated by the force b_1 . From (7.10), we have that

$$\|m(\cdot, \bar{t}_{j'+1}) - \phi_2(\cdot, \bar{t}_{j'+1})\|_{L^2}^2 \leq e^{(2+\beta)S} \frac{\delta_{j'}}{\Delta(\epsilon)}. \quad (8.2)$$

Similarly to (7.11) we have:

$$\frac{d}{dt} \int_{\mathbb{R}} (\phi_2 - m^0)^2(x, t) dx =$$

$$\begin{aligned}
&= -2 \int_{\mathbb{R}} (\phi_2 - m^0)^2(x, t) dx + 2 \int_{\mathbb{R}} b_1(x, t)(\phi_2 - m^0)(x, t) dx \\
&\quad + 2 \int_{\mathbb{R}} (\phi_2 - m^0)(x, t)(\tanh(\beta J * \phi_2(x, t)) - \tanh(\beta J * m^0(x, t))) dx \\
&\leq C \int_{\mathbb{R}} (\phi_2 - m^0)^2(x, t) dx + c \int_{\mathbb{R}} \alpha^2 b_1^2(x, t) dx,
\end{aligned}$$

for c large enough. After applying Gronwall's inequality and (C.1) we obtain:

$$\|\phi_2(\cdot, \bar{t}_{j'+1}) - m^0(\cdot, \bar{t}_{j'+1})\|_{L^2}^2 \leq ce^{(2+\beta)S} \frac{1}{1 - c_*^2 C \Delta(\epsilon)} \delta_{j'}, \quad (8.3)$$

where $\delta_{j'}$ has been defined in (7.8). Combining (8.2) and (8.3), for m constructed in (8.1) we have:

$$\|m(\cdot, \bar{t}_{j'+1}) - m^0(\cdot, \bar{t}_{j'+1})\|_{L^2(\mathbb{R})}^2 \leq ce^{(2+\beta)S} \frac{\delta_{j'}}{\Delta(\epsilon)}. \quad (8.4)$$

Moreover, since by the definition of the time $\bar{t}_{j'+1}$ the profile m is close to \mathcal{M} at that time, we have that

$$\|\bar{m}_{\bar{\xi}(m(\cdot, \bar{t}_{j'+1}))} - \bar{m}_{\bar{\xi}(m^0(\cdot, \bar{t}_{j'+1}))}\|_{L^2(\mathbb{R})}^2 \leq ce^{(2+\beta)S} \frac{\delta_{j'}}{\Delta(\epsilon)} + 7\vartheta, \quad (8.5)$$

for some $c > 0$. From this, we can obtain an estimate for the distance between the centers in $\bar{\xi}(m(\cdot, \bar{t}_{j'+1}))$ and $\bar{\xi}(m^0(\cdot, \bar{t}_{j'+1}))$. Let k be the number of centers of $m(\cdot, t_{j'})$ and $\bar{r}(t_{j'}) = (r_1(t_{j'}), \dots, r_k(t_{j'}))$ with $|r_{i+1}(t_{j'}) - r_i(t_{j'})| \geq |\log \epsilon|^2$, $\forall i$. For $l \in \{1, \dots, k\}$ odd, define i_l to be the odd label such that

$$\min_{i \text{ odd}} |\xi_i - \xi_l^0| = |\xi_{i_l} - \xi_l^0|. \quad (8.6)$$

For l even we define i_l analogously. Furthermore, during the time interval $[t_{j'}, \bar{t}_{j'+1})$, new centers might be created due to nucleations. Let ℓ_1, \dots, ℓ_p be the labels of the newly created centers.

By the properties of the instanton we have that the upper bound in (8.5) induces an upper bound on the volume of the mismatch between $\bar{m}_{\bar{\xi}(m(\cdot, \bar{t}_{j'+1}))}$ and $\bar{m}_{\bar{\xi}(m^0(\cdot, \bar{t}_{j'+1}))}$. Since the centers i_1, \dots, i_k are still far enough, this further induces a bound on the corresponding centers. Hence, both $|\xi_{i_l} - r_l|$ and $|\xi_{\ell_i} - \xi_{\ell_{i+1}}|$, for i odd in $\{1, \dots, k\}$ are bounded by the estimate in (8.5).

In the next iteration, we construct a profile solving (8.1) for $t \geq \bar{t}_{j'+1}$ starting at $m(\bar{t}_{j'+1}, \cdot)$. Using the same argument as before, we choose another time $\bar{t}_{j'+2} \in [j' + 2 - \frac{1}{2}, j' + 2 - \frac{1}{4}]S$ with $m(\bar{t}_{j'+2}, \cdot)$ close to \mathcal{M} . By repeating the same procedure we obtain

$$\|m(\cdot, \bar{t}_{j'+2}) - m^0(\cdot, \bar{t}_{j'+2})\|_{L^2(\mathbb{R})}^2 \leq ce^{(2+\beta)S} \frac{\delta_{j'+1}}{\Delta(\epsilon)}, \quad (8.7)$$

where m^0 is the solution of the equation without the forcing term in the interval $[\bar{t}_{j'+1}, \bar{t}_{j'+2})$ starting at $m(\cdot, \bar{t}_{j'+1})$. This induces a bound on the corresponding centers by the same amount. These could be the original ones, or the ones nucleated in the time interval $[t_{j'}, \bar{t}_{j'+1})$ and continued moving the current one, or those nucleated during the second time interval

$[\bar{t}_{j'+1}, \bar{t}_{j'+2})$. Thus, during the first two bad time intervals of the connected component $[t_{j'}, t_{j''})$, the displacement of the old centers (at time $t_{j'}$) or the distance between the newly created are both bounded by

$$ce^{(2+\beta)S} \frac{\delta_{j'}}{\Delta(\epsilon)} + 7\vartheta + ce^{(2+\beta)S} \frac{\delta_{j'+1}}{\Delta(\epsilon)} + 7\vartheta.$$

At the end of the connected component of the bad time intervals the corresponding estimate is

$$ce^{(2+\beta)S} \frac{1}{\Delta(\epsilon)} \sum_{k=j'}^{j''} \delta_k + \frac{P}{\delta(\epsilon)} 7\vartheta \leq ce^{(2+\beta)S} \frac{P}{\Delta(\epsilon)} + \frac{P}{\delta(\epsilon)} 7\vartheta << |\log \epsilon|^2, \quad (8.8)$$

by the choice in (5.12).

9. THE PARTICLE MODEL, TOTAL COST AND TOTAL DISPLACEMENT

9.1. The “particle” model. Given a profile $\phi \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$, in the previous sections we created a function m with $I(\phi) \geq I(m)$. By construction, see (7.15), at the end of each good time interval the function m has its odd/even centers on the right/left of the corresponding centers of ϕ , eventually after performing a jump by a quantity S_ϵ (see (7.12)), if necessary. To each such center we assign a “particle” whose position is given by the function $t \mapsto r_i(t)$ as defined in (7.5). From (5.3) there is a maximum possible number of such particles, say n^* and we write $\bar{r}(t) := (r_1(t), \dots, r_{n^*}(t))$ for their positions. During a connected component of good time intervals we may have that some of these particles die as a result of a “collision” as described before. On the other hand, during the bad time intervals (where the cost is higher) we may get a birth (or more) of two such particles after the occurrence of a nucleation. Thus, a possible behavior of these particles is the following: at time $t = 0$ we have the particle $r_1(0) = 0$ and $r_i(0) = \emptyset$ for all $2 \leq i \leq n$, which moves in a bad time interval, during which a nucleation takes place at time $t_1^* \geq 0$ and we have the creation of the new particles at positions $r_{i_1}(t_1^*) = r_{i_1+1}(t_1^*)$ (distance $|\log \epsilon|^2$), with i_1 odd (note also that we let $r_{i_1}(t) = r_{i_1+1}(t) = \emptyset$ for $t < t_1^*$). Then the particles enter into a connected component of good time intervals after (possibly) making a jump in their positions r_i by at most $o(|\log \epsilon|^2)$ as shown in Section 8. Then, before entering into the next good time intervals of small cost, new jumps may occur as a result of the initialization described in Section 5. After entering, new jumps have to be taken into account as a result of a jump from a good time interval to the next as in Section 7. In both of these cases (say at a time t_2^*) it may happen that two particles (r_{i_2} and r_{i_2+1}) collapse in which case we write $r_{i_2}(t) = r_{i_2+1}(t) = \emptyset$ for all $t \geq t_2^*$. Hence, following the above rules and the analysis in the previous sections we obtain the configuration of the particles denoted by $\{n, (r_1(t), \dots, r_n(t))\}$ for $t \in [0, \epsilon^{-2}T]$.

9.2. Lower bound. We want to find a lower bound of the total cost determined by the new quantities $\bar{r}(t)$ and the velocities $v_i^0(t)$. Furthermore, we have the constraint that the total displacement is $\geq \epsilon^{-1}R$. From this, we derive a constraint on $v_i^0(t)$, for $t \in [0, \epsilon^{-2}T]$. We have to take into account the displacement during the good time intervals, the jumps S_ϵ^j ,

(7.12), between two good time intervals, the displacement during bad time intervals (8.8) and finally the displacement due to nucleation and collision of particles. Thus, the constraint reads:

$$\begin{aligned} \sum_{i=1}^{n^*} \int_{\{t: r_i(t) \neq \emptyset\}} |v_i^0(t)| \geq & \epsilon^{-1} R - \left(cn^* \sum_{j \in G_{\text{tot}}} \int_{t_j}^{t_{j+1}} (\|\alpha b_1\|_{L^2(d\nu)}^2 + R_{\max}) ds \right. \\ & \left. + c \sum_{j \in G_{\text{tot}}} S_\epsilon^j + |\log \epsilon|^2 + n^* 4 |\log \epsilon|^2 \right). \end{aligned} \quad (9.1)$$

In the good time interval $[t_j, t_{j+1}]$, using (C.2), we have the following lower bound for the cost:

$$\int_{t_j}^{t_{j+1}} \int_{\mathbb{R}} \mathcal{H}(\phi, \dot{\phi})(x, t) dx dt \geq \int_{t_j}^{t_{j+1}} \|\alpha b_1\|_{L^2(d\nu)} dt - \frac{c_*^2 C \Delta(\epsilon)}{1 - c_*^2 C \Delta(\epsilon)} P,$$

where by Hölder's inequality we also have that

$$\|\alpha b_1\|_{L^2(d\nu)} \geq \sum_{i: r_i(t) \neq \emptyset} \left\{ \frac{1}{\|\bar{m}'\|_{L^2(d\nu)}^2} \left| (\alpha b_1, \bar{m}'_{\xi_i(t)})_{L^2(d\nu)} \right| - ce^{-\alpha |\log \epsilon|^2/2} \right\}.$$

Thus, taking also into account the mobility $\mu = 4\|\bar{m}'\|_{L^2(d\nu)}$, in a good time interval we obtain:

$$\int_{t_j}^{t_{j+1}} \int_{\mathbb{R}} \mathcal{H}(\phi, \dot{\phi})(x, t) dx dt \geq \int_{t_j}^{t_{j+1}} \sum_{i: r_i(t) \neq \emptyset} \frac{v_i^0(t)^2}{\mu} - ce^{-\alpha |\log \epsilon|^2/2} 2S - \frac{c_*^2 C \Delta(\epsilon)}{1 - c_*^2 C \Delta(\epsilon)} P.$$

On the other hand, the cost during the bad time intervals (say $[t_0, t_1]$) is neglected unless if a nucleation occurs in which case by reversibility we have that:

$$\int_{t_0}^{t_1} \int_{\mathbb{R}} \mathcal{H}(\phi, \dot{\phi})(x, t) dx dt \geq \mathcal{F}(\phi(\cdot, t_1)) - \mathcal{F}(\phi(\cdot, t_0)).$$

Using (5.5) and the bound n^* on the maximum number of fronts, we have that for the given $\gamma > 0$,

$$\mathcal{F}(\phi(\cdot, t_1)) - \mathcal{F}(\phi(\cdot, t_0)) \geq n^* \mathcal{F}(\bar{m}) - \gamma.$$

Thus, for all $\epsilon > 0$, the total cost is bounded from below by

$$\begin{aligned} \int_0^{\epsilon^{-2}T} \int_{\mathbb{R}} \mathcal{H}(\phi, \dot{\phi})(x, t) dx dt \geq & \int_{G_{\text{tot}}} \sum_{i: r_i(t) \neq \emptyset} \frac{v_i^0(t)^2}{\mu} - ce^{-\alpha |\log \epsilon|^2/2} \epsilon^{-2} T - \frac{c_*^2 C \Delta(\epsilon)}{1 - c_*^2 C \Delta(\epsilon)} P \\ & + 2n^* \mathcal{F}(\bar{m}) - \gamma. \end{aligned} \quad (9.2)$$

Thus, the problem reduces to finding the inf over the velocities $v_i^0(\cdot)$ of the right hand side of (9.2) under the constraint (9.1), where $i = 1, \dots, n^*$ is the index of a front and suppose that its lifetime is T_i . With this estimate, arguing as in [11] we conclude the proof of the lower bound.

9.3. Upper bound. First, we compute the optimal number of nucleations. Then, we construct a sequence $\phi_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$, which at time $t = 0$ consists of a multi-instanton with $2n + 1$ centers at positions 0 and $\frac{2i}{2n+1}\epsilon^{-1}R \pm \frac{1}{2}|\log \epsilon|^2$, for $i = 1, \dots, n$. Then for $t \in (0, \epsilon^{-2}T]$ they move with constant velocity $\frac{V}{2n+1}$ to the right (the odd-numbered) or left (the even-numbered), where $V = R/T$. When they are at a distance smaller than $|\log \epsilon|^2$ they disappear. It is easy to check that this sequence satisfies (2.20).

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APPENDIX A. EXISTENCE OF SOLUTIONS OF THE SYSTEM (5.14)-(5.16)

Recalling the definition of b in (2.9) and of b_1 in (5.13), we define the sequence $\{\tilde{\xi}^k, \phi_1^k, m^k\}_{k \geq 1}$ which solves the following system of equations (for simplicity we work in the good time interval $[0, S]$):

$$b(\phi_1^k) = \alpha_k b_1, \quad \text{with} \quad \phi_1^k(\cdot, 0) = \phi(\cdot, 0) \quad \text{and} \quad (\text{A.1})$$

$$b(m^k) = b(\phi_1^k), \quad \text{with} \quad m^k(\cdot, 0) = m_0(\cdot), \quad (\text{A.2})$$

where

$$\alpha_0 = 1, \quad \alpha_1 = \left(\frac{1 - \bar{m}_{\xi^0}^2}{8} \right)^{\frac{1}{2}} \quad \text{and} \quad \alpha_k = \left(\frac{1 - \bar{m}_{\xi^{k-1}}^2}{8} \right)^{\frac{1}{2}}.$$

The initial condition m_0 is as in the initialization in Section 5 and $\tilde{\xi}^k = (\tilde{\xi}_1^k, \dots, \tilde{\xi}_n^k)$ are the approximate centers of m^k defined as in (6.2). We define the initial center $\tilde{\xi}^0$ as the center of the profile m^0 , defined by:

$$b(m^0) = b_1, \quad \text{with} \quad m^0(\cdot, 0) = m_0(\cdot).$$

Then, m^1 solves the following initial value problem:

$$b(m^1) = \alpha_1 b_1, \quad \text{with} \quad m^1(\cdot, 0) = m_0(\cdot).$$

From the equations above for m^0 and m^1 we have:

$$\frac{d}{dt} \|m^1(\cdot, t) - m^0(\cdot, t)\|_{L^2}^2 \leq (2 + \beta) \|m^1(\cdot, t) - m^0(\cdot, t)\|_{L^2}^2 + \|(1 - \alpha_1)b_1\|_{L^2}^2$$

But, by the definition of c_* in (5.17), it holds that $|(1 - \alpha_k)b_1| \leq c_* \alpha_k |b_1|$, for every $k \geq 1$. Then, applying Gronwall's inequality and using (C.1) we obtain:

$$\|m^1(\cdot, t) - m^0(\cdot, t)\|_{L^2}^2 \leq c e^{(2+\beta)S} \frac{1}{1 - c_*^2 C \Delta(\epsilon)} \int_{\mathbb{R}} \int_0^t \mathcal{H}(x, s) ds dx \leq c e^{(2+\beta)S} \frac{1}{1 - c_*^2 C \Delta(\epsilon)} \delta(\epsilon), \quad (\text{A.3})$$

for some new constant $c > 0$. We define

$$\|\tilde{\xi}^k - \tilde{\xi}^{k-1}\| := \max_{i=1, \dots, n} |\tilde{\xi}_i^k - \tilde{\xi}_i^{k-1}| \quad (\text{A.4})$$

and estimate $|\tilde{\xi}_i^1 - \tilde{\xi}_i^0|$, for $i \in \{1, \dots, n\}$ by

$$|\tilde{\xi}_i^1 - \tilde{\xi}_i^0| \leq c \|m^1 - m^0\|_{L^2}.$$

We first show that $\{\tilde{\xi}^k\}_{k \geq 0} \subset L^\infty([0, S]; \mathbb{R}^n)$ is a Cauchy sequence. By following the same reasoning as in (A.3), for every $k \geq 1$ we have that

$$\begin{aligned} \|m^k(\cdot, t) - m^{k-1}(\cdot, t)\|_{L^2}^2 &\leq c e^{(2+\beta)S} \int_0^t \|b(m^k)(\cdot, s) - b(m^{k-1})(\cdot, s)\|_{L^2}^2 ds \\ &\leq c e^{(2+\beta)S} 2 \frac{1}{1 - c_*^2 C \Delta(\epsilon)} \delta(\epsilon). \end{aligned} \quad (\text{A.5})$$

Therefore, since $\|m^k - m^{k-1}\|_{L^2}$ is small, given a mixed contour Γ_i we have that:

$$|\tilde{\xi}_i^k - \tilde{\xi}_i^{k-1}| \leq C \|m^k - m^{k-1}\|_{L^2}. \quad (\text{A.6})$$

For the difference between the two forces $b(m^k)$ and $b(m^{k-1})$, from (A.1) and (A.2) we have:

$$\begin{aligned} \int_0^t \|b(m^k) - b(m^{k-1})\|_{L^2}^2 ds &= \int_0^t \int_{\mathbb{R}} \left(\left(\frac{1 - \bar{m}_{\tilde{\xi}^{k-1}}^2}{8} \right)^{\frac{1}{2}} - \left(\frac{1 - \bar{m}_{\tilde{\xi}^{k-2}}^2}{8} \right)^{\frac{1}{2}} \right)^2 b_1(x, s)^2 dx ds \\ &\leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} |\bar{m}_{\tilde{\xi}^{k-1}}^2 - \bar{m}_{\tilde{\xi}^{k-2}}^2| b_1(x, s)^2 dx ds \\ &\leq \frac{(\Delta(\epsilon))^2}{4} \sum_{i=1}^n \int_0^t \int_{\Gamma_i} |\bar{m}_{\tilde{\xi}^{k-1}} - \bar{m}_{\tilde{\xi}^{k-2}}| \mathbf{1}_{[|b(\phi)| \leq \Delta(\epsilon)]} dx ds \\ &\leq \frac{(\Delta(\epsilon))^2}{2} nS \|\bar{m}'\|_{L^1} \sup_{0 \leq s \leq t} \|\tilde{\xi}^{k-1} - \tilde{\xi}^{k-2}\|(s) \end{aligned} \quad (\text{A.7})$$

In the above computations we exploited the fact that m^k and m^{k-1} have the same number of contours and their centers are close to each other due to (A.6). We combine (A.5), (A.6), (A.7) and for ϵ sufficiently small we obtain a contraction:

$$\sup_t \|\tilde{\xi}^k - \tilde{\xi}^{k-1}\| \leq L \sup_t \|\tilde{\xi}^{k-1} - \tilde{\xi}^{k-2}\|$$

where $L = C \|\bar{m}'\|_{L^1} e^{\beta S} \Delta^2 nS < 1$.

Similarly, using the same estimates we can show that the sequences $\{m^k\}_k$ and $\{\phi_1^k\}_k$ are Cauchy in the norm $\sup_t (\|\cdot\|_{W^{1,1}})$ and using a standard argument we can show that the limit point satisfies the system.

APPENDIX B. L^1 AND L^2 BOUNDS ON THE CENTERS

We denote

$$\mathcal{N} = \{m \in L^\infty(\mathbb{R}, [-1, 1]) : \limsup_{x \rightarrow -\infty} m(x) < 0; \liminf_{x \rightarrow +\infty} m(x) > 0\}$$

and define the δ neighborhood of $\mathcal{M}^{(1)} := \{\bar{m}_\xi, \xi \in \mathbb{R}\}$ by

$$\mathcal{M}_\delta^{(1)} = \bigsqcup_{\xi \in \mathbb{R}} \{m \in L^\infty(\mathbb{R}, [-1, 1]) : \|m - \bar{m}_\xi\|_{L^2} < \delta\}.$$

Lemma B.1. *Any $m \in \mathcal{N}$ has a center. Moreover, there are positive constants c and δ so that any $m \in \mathcal{M}_\delta^{(1)}$ has a unique center $\xi(m)$. Furthermore, for any $n \in \mathcal{M}_\delta^{(1)}$ with $\|m - n\|_{L^1}$ small we have:*

$$|\xi(m) - \xi(n)| \leq c\|m - n\|_{L^1}.$$

The same result also holds for the $\|\cdot\|_{L^2}$ norm.

Proof: From the definition of a center it suffices to find a ξ such that

$$(m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} = 0 \tag{B.1}$$

The function $\xi \mapsto (m, \bar{m}'_\xi)_{L^2(d\nu_\xi)}$ is a continuous function and by the definition of \mathcal{N} we have that

$$\limsup_{x \rightarrow -\infty} (m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} < 0; \liminf_{x \rightarrow +\infty} (m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} > 0.$$

Thus (B.1) has a solution.

To show uniqueness, since the function m is in the δ -ball around some \bar{m}_{ξ_0} (without loss of generality we can also assume that $\xi_0 = 0$), we write

$$m = \bar{m} + \psi, \quad \|\psi\|_{L^2(d\nu)} < \delta.$$

Then (B.1) gives $(m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} = -(\psi, \bar{m}'_\xi)_{L^2(d\nu_\xi)}$ and since $\|\psi\|_{L^2(d\nu)} < \delta$, we obtain that

$$|(\psi, \bar{m}'_\xi)_{L^2(d\nu_\xi)}| \leq \|\psi\|_{L^2(d\nu)} \|\bar{m}'_\xi\|_{L^2(d\nu_\xi)} \leq \delta \frac{1 - \bar{m}_\xi^2}{1 - \bar{m}_\xi^2} \|\bar{m}'_\xi\|_{L^2(d\nu_\xi)}, \quad \text{for any } \xi \in \mathbb{R}. \tag{B.2}$$

Following [24], Theorem 8.5.1.1, we choose $\delta < \frac{\alpha_0}{\|\bar{m}'_\xi\|_{L^2(d\nu_\xi)}}$ which implies that there is no solution to (B.2) when $|\xi| \geq 1$ and $\|m - \bar{m}\|_{L^2(d\nu)} < \delta$.

Given n with $\|m - n\|_{L^1}$ small, we write: $n = m + \chi$, with $\|\chi\|_{L^1} < \delta'$. We define

$$g(\xi) := (\bar{m}, \bar{m}'_\xi)_{L^2(d\nu_\xi)} + (\psi, \bar{m}'_\xi)_{L^2(d\nu_\xi)} + (\chi, \bar{m}'_\xi)_{L^2(d\nu_\xi)} \tag{B.3}$$

Then $\xi(n)$ is defined by $g(\xi(n)) = 0$. We have:

$$0 = g(\xi(n)) = (\chi, \bar{m}'_{\xi(m)})_{L^2(d\nu_{\xi(m)})} + \int_{\xi(m)}^{\xi(n)} g'(z) dz$$

Since $|\xi(n)| \leq 1$ and $|\xi(m)| \leq 1$ we have that $|z| \leq 1$, thus $g'(z) \geq \alpha_0/2$. Hence,

$$|\xi(n) - \xi(m)| \leq \frac{2}{\alpha_0} |(\chi, \bar{m}'_{\xi(m)})_{L^2(d\nu_{\xi(m)})}| \leq \frac{2}{\alpha_0} \|\chi\|_{L^1} \left\| \frac{\bar{m}'_{\xi(m)}}{1 - \bar{m}_{\xi(m)}^2} \right\|_{\infty}$$

which concludes the proof. Alternatively, we can have the following inequality:

$$|\xi(n) - \xi(m)| \leq \frac{2}{\alpha_0} |(\chi, \bar{m}'_{\xi(m)})_{L^2(d\nu_{\xi(m)})}| \leq \frac{2}{\alpha_0} \|\chi\|_{L^2(d\nu)} \|\bar{m}'_{\xi(m)}\|_{L^2(dx)},$$

which concludes the proof for the case of the L^2 norm as well. \square

APPENDIX C. ASYMPTOTIC ANALYSIS OF \mathcal{H}

For \mathcal{H} given in (2.13) we have that uniformly on $u \in [-1, 1]$ and $w \in (-1, 1)$:

$$\lim_{|b| \rightarrow \infty} \frac{\mathcal{H}(b, u, w)}{|b| \log(|b| + 1)} = \frac{1}{2} \quad \text{and} \quad \lim_{|b| \rightarrow 0} \frac{\mathcal{H}(b, u, w)}{b^2} = \frac{1}{4(1 + uw)}.$$

Moreover, for the choice of $\Delta(\epsilon)$ in (5.12), in the case $|b| \leq \Delta(\epsilon)$, we have that:

$$|\mathcal{H}(b, u, w) - \frac{1}{4(1 + uw)} b^2| \leq C |b|^3 \leq C \Delta(\epsilon)^3,$$

for some $C > 0$. Thus, for b_1 defined in (5.13), using (5.17) we have that for the same constant $C > 0$ the following hold:

$$\int_{\{|b| \leq \Delta(\epsilon)\}} |\alpha(x, t) b_1(x, t)|^2 dx dt \leq \frac{1}{1 - c_*^2 C \Delta(\epsilon)} \int_{\{|b| \leq \Delta(\epsilon)\}} \mathcal{H}(b, u, w) dx dt \quad (\text{C.1})$$

and

$$\begin{aligned} \int_{\{|b| \leq \Delta(\epsilon)\}} \left| \mathcal{H}(b, u, w) - \frac{1}{4(1 + uw)} b_1^2 \right| dx dt &\leq C \Delta(\epsilon) \int_{\{|b| \leq \Delta(\epsilon)\}} b^2(x, t) dx dt \\ &\leq c_*^2 C \Delta(\epsilon) \int_{\{|b| \leq \Delta(\epsilon)\}} |\alpha(x, t) b(x, t)|^2 dx dt. \end{aligned}$$

Adding and subtracting $\int_{\{|b| \leq \Delta(\epsilon)\}} \mathcal{H}(b, u, w) dx dt$, for ϵ small enough it is further implied that

$$\int_{\{|b| \leq \Delta(\epsilon)\}} \left| \mathcal{H}(b, u, w) - \frac{1}{4(1 + uw)} b_1^2 \right| dx dt \leq \frac{c_*^2 C \Delta(\epsilon)}{1 - c_*^2 C \Delta(\epsilon)} \int_{\{|b| \leq \Delta(\epsilon)\}} \mathcal{H}(b, u, w) dx dt, \quad (\text{C.2})$$

which is small as $\epsilon \rightarrow 0$ since the cost is bounded by P and $\Delta(\epsilon) \rightarrow 0$.

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